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SYMMETRY AND REDUCTION IN IMPLICIT GENERALIZED HAMILTONIAN SYSTEMS

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In this paper we study the notion of symmetry for implicit generalized Hamiltonian systems, which are Hamiltonian systems with respect to a generalized Dirac structure. We investigate the reduction of these systems admitting a symmetry Lie group with corresponding conserved quantities. Main features in this approach concern the projection and restriction of Dirac structures, generalizing the corresponding theory for symplectic forms and Poisson brackets. The results are applied to the theory of symmetries and reduction in nonholonomically constrained mechanical systems. The main result extends the reduction theory for explicit Hamiltonian systems and constrained mechanical systems to a general unified reduction theory for implicit generalized Hamiltonian systems.

Keywords: constraints, Dirac structures, Hamiltonian systems, implicit systems, reduction, symmetry.

1. Introduction

The classical result by Noether, stating that to a symmetry of a mechanical system there corresponds a conserved quantity (or more general, to a symmetry Lie group of the system there corresponds a set of conserved quantities, classically called the momentum map), has been very important for the reduction of mechanical systems. The result implies that the mechanical system, after reduction by factoring out the symmetry, can be further reduced to a level set of the conserved quantity, resulting in a reduction of order two, corresponding to the one-dimensional symmetry (group). This fact has been very important for instance for the theory of completely integrable systems, such as the rigid body where the symmetry group is given by the group of Euclidean motions in the three-dimensional space, $SE(3)$, and the corresponding conserved quantities are given by the angular momentum (corresponding to rotation) as well as the linear momentum (corresponding to translation). The theory of Noether has been generalized to general Hamiltonian systems (that is, not necessarily defined on the cotangent bundle of the configuration space), resulting in the well-known symplectic reduction theorem of Marsden and Weinstein [16], see also [1], which was later generalized to the Poisson case by Marsden and Ratiu [15]. This reduction

theory is not only important for the actual solving of the equations of motion (as for instance in the completely integrable case), but also for the stability analysis of these Hamiltonian systems, see e.g. [21, 1].

Lately there has been much interest in mechanical systems with nonholonomic kinematic constraints. Examples of such systems are given by the rolling penny and the snakeboard, where the nonholonomic constraints are given by non-slipping conditions, see for example [20, 7]. Such systems also admit certain symmetry groups, such as $SE(2)$ for the two examples mentioned. There has been a lot of literature describing the reduction possibilities of these systems subject to symmetry groups, we only mention [3, 7, 8, 22, 23] and refer to the references therein. The basic difference with the unconstrained case is that Noether's relation between symmetries and conserved quantities in general does not hold anymore, that is, symmetries do not necessarily give rise to conserved quantities. This of course has important consequences for the possible order of reduction of these systems. However, if one considers horizontal symmetries, i.e. symmetries that are *consistent* with the constraints, one can show the existence of corresponding conserved quantities, see the references above. In this case one can again use these conserved quantities to further reduce the system to a lower dimensional nonholonomically constrained mechanical system, see e.g. [22, 8].

In the present paper we will extend the reduction theory for explicit Hamiltonian systems and constrained mechanical systems, as described above, to a general reduction theory for the so-called implicit generalized Hamiltonian systems. These systems are defined using the geometric notion of a generalized Dirac structure, as introduced in [9, 11] as a generalization of the classical notions of symplectic and Poisson structures. Implicit generalized Hamiltonian systems describe in general a mixed set of differential and algebraic equations (DAE's), and have shown to be instrumental in the description of energy conserving physical systems, as well as power conserving interconnections of these systems, see [10, 24–26]. Examples of these systems not only include nonholonomically constrained mechanical systems but also electrical systems such as LC-networks [6] and electromechanical systems [24].

Investigation of the reduction possibilities of implicit generalized Hamiltonian systems was started in [23], where it was shown that an implicit generalized Hamiltonian system admitting a symmetry Lie group can be reduced to a lower dimensional implicit generalized Hamiltonian system by factoring out this group. In this paper we continue up on these results. We shall prove that an implicit generalized Hamiltonian system having a conserved quantity (called first integral) can be reduced to a system on a level set of this conserved quantity. The important observation is that this system is again an implicit generalized Hamiltonian system. Furthermore we define horizontal symmetries for these systems, and we show that these symmetries give rise to conserved quantities, thereby adding the “second step” of the reduction theory. Finally, we show that starting with (horizontal) symmetries and corresponding conserved quantities, it does not make a difference if one starts reducing the system by restriction to the level set of the conserved quantities and then factoring out the (residual) symmetry group, or first factoring out the symmetry group and then re-

stricting to the level set of the remaining conserved quantities (which will now be *Casimir* functions). This result is a generalization of the reduction theory described for instance in [13, 19], thereby regaining the full classical reduction picture in the setting of implicit generalized Hamiltonian systems.

The paper is organized as follows. In Section 2 we give an introduction to Dirac structures and implicit generalized Hamiltonian systems. We introduce some basic notions and results and give some examples, including the description of constrained mechanical systems. In Section 3 the notion of a symmetry of an implicit generalized Hamiltonian system is investigated. We recall some important results obtained in [23] and derive some new ones. Furthermore, the notions of first integral (or conserved quantity) and Casimir function, which are important for the reduction process described in Sections 4 and 5, are introduced. In Section 4 the basic results on reduction of Dirac structures and implicit generalized Hamiltonian systems, consisting of reducing the system to a level set of a first integral or factoring out a symmetry Lie group (recalling the result in [23] and giving an extended proof), are derived. These results are combined in Section 5 to derive our main result on reduction of implicit generalized Hamiltonian systems, admitting symmetries with corresponding conserved quantities. We show that reducing the system by restriction to the level set of the conserved quantities and then factoring out the (residual) symmetry group, or first factoring out the symmetry group and then restricting to the level set of the remaining conserved quantities will result in the same (up to isomorphism) implicit generalized Hamiltonian system. This result will generalize the classical reduction theorems of explicit Hamiltonian systems described in [16, 1, 13, 15, 19]. Section 6 gives a proof of a result used in Section 5. In Section 7 the main reduction result of Section 5 is specialized to implicit generalized Hamiltonian systems satisfying an additional regularity assumption on the constraints, which makes the system explicit in some sense. The reduction result in this case is compared with the result in the classical explicit case. Finally, in Section 8 the theory is connected to the theory of symmetries and reduction in constrained mechanical systems. We define the notion of a horizontal symmetry and we show that these symmetries give rise to first integrals. This will lead to the same conclusion as in [22].

We refer to [5] for some more details. Some of the results described here have already appeared in abridged form in [4].

2. Implicit generalized Hamiltonian systems

In this section we will give an introduction to Dirac structures and implicit generalized Hamiltonian systems. For more information we refer to [26, 17, 23, 9, 11]. Let \mathcal{X} be an n -dimensional manifold with tangent bundle $T\mathcal{X}$ and cotangent bundle $T^*\mathcal{X}$. Define $T\mathcal{X} \oplus T^*\mathcal{X}$ as the smooth vector bundle over \mathcal{X} with fibre at each $x \in \mathcal{X}$ given by $T_x\mathcal{X} \times T_x^*\mathcal{X}$. Let X be a smooth vector field and α a smooth one-form on \mathcal{X} , respectively. We say that the pair (X, α) belongs to a subspace $D \subset T\mathcal{X} \oplus T^*\mathcal{X}$, denoted $(X, \alpha) \in D$, if $(X(x), \alpha(x)) \in D(x)$, $\forall x \in \mathcal{X}$. In the

sequel we will not make a notational distinction between the tangent bundle $T\mathcal{X}$ and the ring of smooth sections of $T\mathcal{X}$. The same holds for the cotangent bundle $T^*\mathcal{X}$. That is, $X \in T\mathcal{X}$ will always denote a globally defined smooth vector field on \mathcal{X} , and $\alpha \in T^*\mathcal{X}$ will always denote a globally defined smooth one-form on \mathcal{X} . So D will denote a set of pairs (X, α) , with X a smooth vector field and α a smooth one-form on \mathcal{X} .

Let D be a linear subspace of $T\mathcal{X} \oplus T^*\mathcal{X}$, that is, $(X, \alpha), (Y, \beta) \in D$ implies $h_1(X, \alpha) + h_2(Y, \beta) \in D$ for all $h_1, h_2 \in C^\infty(\mathcal{X})$. Define the linear subspace D^\perp as

$$D^\perp = \{(Y, \beta) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \langle \alpha, Y \rangle + \langle \beta, X \rangle = 0, \forall (X, \alpha) \in D\},$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between a one-form and a vector field on \mathcal{X} .

DEFINITION 1 [11, 9, 10]. A *generalized Dirac structure* on \mathcal{X} is a linear subspace $D \subset T\mathcal{X} \oplus T^*\mathcal{X}$ such that $D = D^\perp$.

From the condition $D = D^\perp$ it follows that D is constant dimensional, with $\dim D(x) = n$, $\forall x \in \mathcal{X}$, see also [11], i.e. D is a subbundle of $T\mathcal{X} \oplus T^*\mathcal{X}$. This has the following obvious but important consequence.

PROPOSITION 1. Let D be a generalized Dirac structure. Then $D^\perp(x) = [D(x)]^\perp$, $\forall x \in \mathcal{X}$. Here $[D(x)]^\perp$ means the pointwise perpendicular to $D(x)$, i.e.

$$[D(x)]^\perp = \{(w, w^*) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid \langle v^*, w \rangle + \langle w^*, v \rangle = 0, \forall (v, v^*) \in D(x)\}.$$

Proof: It immediately follows that $D^\perp(x) \subset [D(x)]^\perp$. Both $D^\perp(x)$ and $[D(x)]^\perp$ are linear (over \mathbb{R}) subspaces of $T_x\mathcal{X} \times T_x^*\mathcal{X}$. Furthermore, since $\dim D(x) = n$ ($= \dim D^\perp(x)$ since $D = D^\perp$) it follows that $\dim [D(x)]^\perp = n$. This implies that $D^\perp(x) = [D(x)]^\perp$. \square

Since $D = D^\perp$ it immediately follows that for every pair $(X, \alpha) \in D$

$$\langle \alpha, X \rangle = 0. \quad (1)$$

A generalized Dirac structure is called *closed*, or just a Dirac structure, if the following condition holds.

DEFINITION 2. A generalized Dirac structure D on an n -dimensional manifold \mathcal{X} is called *closed* if

$$\langle L_{X_1}\alpha_2, X_3 \rangle + \langle L_{X_2}\alpha_3, X_1 \rangle + \langle L_{X_3}\alpha_1, X_2 \rangle = 0,$$

for all pairs $(X_1, \alpha_1), (X_2, \alpha_2)$ and (X_3, α_3) in D .

Here $L_X\alpha$ denotes the Lie derivative of a one-form α with respect to a vector field X . We have the following theorem.

THEOREM 2 [9–11]. A generalized Dirac structure D on \mathcal{X} is closed if and only if

$$([X_1, X_2], i_{X_1}d\alpha_2 - i_{X_2}d\alpha_1 + d\langle \alpha_2, X_1 \rangle) \in D, \quad \forall (X_1, \alpha_1), (X_2, \alpha_2) \in D.$$

EXAMPLE 1. Let ω be a nondegenerate two-form on \mathcal{X} , then

$$D = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \alpha = i_X \omega\}$$

is a generalized Dirac structure on \mathcal{X} . D is closed if and only if $d\omega = 0$. This corresponds to a symplectic structure (\mathcal{X}, ω) .

EXAMPLE 2. Let $J(x) : T_x^*\mathcal{X} \rightarrow T_x\mathcal{X}$, $x \in \mathcal{X}$, be a skew-symmetric vector bundle map, then

$$D = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid X(x) = J(x)\alpha(x), \quad \forall x \in \mathcal{X}\}$$

is a generalized Dirac structure on \mathcal{X} . This corresponds to a Poisson structure $(\mathcal{X}, \{\cdot, \cdot\})$, where $J(x)$ is the structure matrix of the Poisson bracket $\{\cdot, \cdot\}$. D is closed if and only if the bracket satisfies the Jacobi identity.

Examples 1 and 2 show that the notion of a (generalized) Dirac structure is a generalization of the classical symplectic and Poisson structures.

Corresponding to a generalized Dirac structure D on \mathcal{X} we define the following (co-)distributions

$$\begin{aligned} G_0 &= \{X \in T\mathcal{X} \mid (X, 0) \in D\}, \\ G_1 &= \{X \in T\mathcal{X} \mid \exists \alpha \in T^*\mathcal{X} \text{ such that } (X, \alpha) \in D\}, \\ P_0 &= \{\alpha \in T^*\mathcal{X} \mid (0, \alpha) \in D\}, \\ P_1 &= \{\alpha \in T^*\mathcal{X} \mid \exists X \in T\mathcal{X} \text{ such that } (X, \alpha) \in D\}. \end{aligned}$$

Define the annihilator of a smooth distribution $L \subset T\mathcal{X}$ as the smooth codistribution

$$\text{ann } L = \{\alpha \in T^*\mathcal{X} \mid \langle \alpha, X \rangle = 0, \quad \forall X \in L\},$$

and the kernel of a smooth codistribution $K \subset T^*\mathcal{X}$ as the smooth distribution

$$\ker K = \{X \in T\mathcal{X} \mid \langle \alpha, X \rangle = 0, \quad \forall \alpha \in K\}.$$

Note that the annihilator and the kernel are defined in terms of their smooth sections. That is, the annihilator $\text{ann } L$ consists of all globally defined smooth one-forms α satisfying $\langle \alpha, X \rangle = 0$ for all globally defined smooth vector fields $X \in L$, and analogously for the kernel. It follows that by definition $G_0 = \ker P_1$ and $P_0 = \text{ann } G_1$. Furthermore, we have that $P_1 \subset \text{ann } G_0$ and $G_1 \subset \ker P_0$, with equality if and only if P_1 , respectively G_1 , is constant dimensional [10]. From Theorem 2 it follows that G_0, G_1 and P_1 are involutive if D is closed (if G_1 is constant dimensional it follows that also P_0 is involutive).

We have the following two important representations of a generalized Dirac structure.

THEOREM 3 [10]. Let D be a generalized Dirac structure on a manifold \mathcal{X} .

- (a) If G_1 is constant dimensional, then there exists a skew-symmetric linear map $\omega(x) : G_1(x) \subset T_x\mathcal{X} \rightarrow (G_1(x))^* \subset T_x^*\mathcal{X}$, $x \in \mathcal{X}$, with kernel G_0 , such that

$$D = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \alpha(x) - \omega(x)X(x) \in \text{ann } G_1(x), \forall x \in \mathcal{X}, X \in G_1\}. \quad (2)$$

- (b) If P_1 is constant dimensional, then there exists a skew-symmetric linear map $J(x) : P_1(x) \subset T_x^*\mathcal{X} \rightarrow (P_1(x))^* \subset T_x\mathcal{X}$, $x \in \mathcal{X}$, with kernel P_0 , such that

$$D = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid X(x) - J(x)\alpha(x) \in \ker P_1(x), \forall x \in \mathcal{X}, \alpha \in P_1\}. \quad (3)$$

Conversely, if D is defined as in (2) for some skew-symmetric linear map $\omega(x) : T_x\mathcal{X} \rightarrow T_x^*\mathcal{X}$, $x \in \mathcal{X}$, and constant dimensional distribution $G_1 \subset T\mathcal{X}$, respectively if D is defined as in (3) for some skew-symmetric linear map $J(x) : T_x^*\mathcal{X} \rightarrow T_x\mathcal{X}$, $x \in \mathcal{X}$, and constant dimensional codistribution $P_1 \subset T^*\mathcal{X}$, then D is a generalized Dirac structure on \mathcal{X} .

Note that if $G_1 = T\mathcal{X}$ and $G_0 = 0$, then we are in the situation of Example 1, whereas if $P_1 = T^*\mathcal{X}$, then we are in the situation of Example 2.

The set of admissible functions corresponding to a generalized Dirac structure D is defined as

$$\mathcal{A}_D = \{H \in C^\infty(\mathcal{X}) \mid dH \in P_1\}.$$

There is a well-defined generalized Poisson bracket on \mathcal{A}_D given by [10]

$$\{H_1, H_2\}_D = \langle dH_1, X_2 \rangle = -\langle dH_2, X_1 \rangle,$$

where $H_1, H_2 \in \mathcal{A}_D$, i.e. $(X_1, dH_1), (X_2, dH_2) \in D$. In [10] it is shown that if D is closed, then $\{\cdot, \cdot\}_D$ becomes a true Poisson bracket and turns \mathcal{A}_D into a Lie algebra.

Now we will define the notion of an implicit generalized Hamiltonian system.

DEFINITION 3 [10]. Let D be a (generalized) Dirac structure on a manifold \mathcal{X} . Let $H \in C^\infty(\mathcal{X})$ be a smooth function on \mathcal{X} , called the Hamiltonian or energy function. Then the implicit (generalized) Hamiltonian system corresponding to (\mathcal{X}, D, H) is defined by the specification

$$(\dot{x}, dH(x)) \in D(x), \quad x \in \mathcal{X}.$$

Usually we will denote the implicit (generalized) Hamiltonian system by the triple (\mathcal{X}, D, H) .

EXAMPLE 3. Consider the generalized Dirac structure given in Example 1, then the corresponding implicit generalized Hamiltonian system is precisely the classical Hamiltonian system defined by the two-form ω

$$dH = \omega(X_H, \cdot), \quad (4)$$

where X_H is the vector field corresponding to the solution $x(t)$, i.e. $\dot{x} = X_H(x)$. D is closed if and only if there exist local coordinates (q, p) for x for which the system (4) for an arbitrary Hamiltonian H takes the form

$$\dot{q} = \frac{\partial H}{\partial p}(q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p),$$

which are just the classical canonical Hamiltonian equations.

EXAMPLE 4. Consider the generalized Dirac structure given in Example 2, then the corresponding implicit generalized Hamiltonian system is given by

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x). \quad (5)$$

This is precisely the classical Hamiltonian dynamics given by the Poisson bracket, i.e. $\dot{x} = \{x, H\}$. Again, D is closed if and only if there exist local coordinates (q, p, r) for x for which (5) for an arbitrary Hamiltonian H takes the form

$$\dot{q} = \frac{\partial H}{\partial p}(q, p, r), \quad \dot{p} = -\frac{\partial H}{\partial q}(q, p, r), \quad \dot{r} = 0. \quad (6)$$

Let us reflect on Definition 3 a bit more. First we will define the concept of a solution of the implicit (generalized) Hamiltonian system (\mathcal{X}, D, H) .

DEFINITION 4. A solution of the implicit (generalized) Hamiltonian system (\mathcal{X}, D, H) is defined as a smooth time function $x : I \subset \mathbb{R} \rightarrow \mathcal{X}$ such that

$$(X_H, dH)(x(t)) \in D(x(t)), \quad \forall t \in I,$$

where $X_H(x(t)) = \dot{x}(t)$, $\forall t \in I$, and where I is the interval of existence of $x(t)$, i.e. the domain of x .

It follows from (1) that we have the usual invariance of the Hamiltonian, or conservation of energy, along solutions

$$\frac{dH}{dt}(x(t)) = \langle dH(x(t)), X_H(x(t)) \rangle = 0, \quad \forall t \in I.$$

In general, the implicit generalized Hamiltonian system (\mathcal{X}, D, H) defines a mixed set of differential and algebraic equations (DAE's). Take for instance the Dirac structure given in (3). The corresponding implicit generalized Hamiltonian system (\mathcal{X}, D, H) , for any $H \in C^\infty(\mathcal{X})$, is given by

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)\lambda, \quad (7)$$

$$0 = g^T(x) \frac{\partial H}{\partial x}(x), \quad (8)$$

where $g(x)$ is any full rank matrix such that $\text{Im } g(x) = G_0(x) = \ker P_1(x)$. Eqs. (7), (8) define a set of DAE's, where the algebraic equations are given by (8). The variables λ can be seen as Lagrange multipliers, required to keep the constraint equations (8) to be satisfied for all time.

In general, define the *constraint manifold* (corresponding to an implicit generalized Hamiltonian system (\mathcal{X}, D, H))

$$\mathcal{X}_c = \{x \in \mathcal{X} \mid dH(x) \in P_1(x)\}.$$

Then it follows that every solution $x(t)$ of (\mathcal{X}, D, H) necessarily is contained in \mathcal{X}_c . Notice that not through every point of \mathcal{X}_c there has to go a solution of (\mathcal{X}, D, H) . Also notice that in general the solutions of (\mathcal{X}, D, H) are not unique. This happens for instance if the Lagrange multipliers λ in (7), (8) are not uniquely determined. If λ is uniquely determined, then the solutions of (\mathcal{X}, D, H) are unique. This is the case when the implicit generalized Hamiltonian system (\mathcal{X}, D, H) satisfies Assumption 4 (see further). In that case there goes through every point $x_c \in \mathcal{X}_c$ a unique solution $x(t)$ of (\mathcal{X}, D, H) , see Proposition 5.

An implicit generalized Hamiltonian system (\mathcal{X}, D, H) can be reduced to an *explicit* generalized Hamiltonian system on \mathcal{X}_c provided the following assumption is satisfied. Systems satisfying this assumption are called implicit generalized Hamiltonian systems with index 1.

ASSUMPTION 4. Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) , with D a generalized Dirac structure on \mathcal{X} . Assume that P_1 is constant dimensional, so that D can be represented as in Theorem 3(b). Let $G_0(x) = \text{Im } g(x) = \text{span}\{g_1(x), \dots, g_m(x)\}$, with g_1, \dots, g_m linearly independent vector fields on \mathcal{X} (note that $G_0 = \ker P_1$ is constant dimensional because P_1 is constant dimensional). Assume that the $m \times m$ matrix $[L_{g_i} L_{g_j} H(x)]_{i,j=1,\dots,m}$ is invertible for all $x \in \mathcal{X}_c$.

PROPOSITION 5 [23]. Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) and let Assumption 4 be satisfied. Then (\mathcal{X}, D, H) reduces to an explicit generalized Hamiltonian system on \mathcal{X}_c , denoted by $(\mathcal{X}_c, D_c, H_c)$, given by

$$\dot{x}_c = J_c(x_c) \frac{\partial H_c}{\partial x_c}(x_c) =: X_{H_c}(x_c), \quad (9)$$

where $x_c \in \mathcal{X}_c$, $J_c(x_c) : T_{x_c}^* \mathcal{X}_c \rightarrow T_{x_c} \mathcal{X}_c$ is a skew-symmetric vector bundle map, and $H_c : \mathcal{X}_c \rightarrow \mathbb{R}$ denotes the restriction of H to \mathcal{X}_c .

We refer to [23] for the actual construction of the vector bundle map J_c . Actually, it can be shown that $(\mathcal{X}_c, D_c, H_c)$ equals the reduction of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) to the submanifold $\mathcal{X}_c \subset \mathcal{X}$, as described in Section 4.2 (see also Remark 5). Proposition 5 becomes very transparent if we consider an implicit Hamiltonian system (\mathcal{X}, D, H) , i.e. with a generalized Dirac

structure D which is closed. Then around every point $x \in \mathcal{X}$ there exist local coordinates (q, p, r, s) for which the system (\mathcal{X}, D, H) takes the form

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}(q, p, r, s), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p, r, s), \\ \dot{r} &= 0, \\ 0 &= \frac{\partial H}{\partial s}(q, p, r, s),\end{aligned}$$

see [10]. Assuming that the system (\mathcal{X}, D, H) satisfies Assumption 4 is equivalent in this case to assuming that the matrix $\frac{\partial^2 H}{\partial s^2}(q, p, r, s)$ is nonsingular. Hence by the implicit function theorem we can locally express s in the coordinates q, p, r , that is, $s = s(q, p, r)$. Defining the constrained Hamiltonian $H_c(q, p, r) = H(q, p, r, s(q, p, r))$, the implicit Hamiltonian system (\mathcal{X}, D, H) becomes the *explicit* Hamiltonian system

$$\begin{aligned}\dot{q} &= \frac{\partial H_c}{\partial p}(q, p, r), \\ \dot{p} &= -\frac{\partial H_c}{\partial q}(q, p, r), \\ \dot{r} &= 0.\end{aligned}$$

As a final example we will describe constrained mechanical systems in this setting.

EXAMPLE 5 [23]. Consider a mechanical system with configuration manifold Q . The phase (or state) space T^*Q is endowed with the canonical symplectic form ω . Describe the linear nonholonomic constraints by a set $\{\alpha_1, \dots, \alpha_k\}$ of independent one-forms on Q , and (vertically) lift these constraints to constraints on T^*Q , defining the codistribution $P_0 = \text{ann } G_1 = \text{span}\{\pi^*\alpha_1, \dots, \pi^*\alpha_k\}$, where $\pi : T^*Q \rightarrow Q$ is the natural projection. Then D defined as in (2) defines a generalized Dirac structure on T^*Q . Let there be given a Hamiltonian function $H : T^*Q \rightarrow \mathbb{R}$, representing the total energy (= kinetic plus potential energy) in the system. Then (T^*Q, D, H) describes a nonholonomically constrained mechanical system, as investigated for instance in [3]. If one takes local canonical coordinates (q, p) for T^*Q and expresses the one-forms $\alpha_1, \dots, \alpha_k$ locally by the rows of the matrix $A^T(q)$, one gets the system

$$\begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} + \begin{bmatrix} A(q) \\ 0 \end{bmatrix} \lambda,$$

$$0 = \begin{bmatrix} A^T(q) & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix},$$

where λ are the Lagrange multipliers representing the constraint forces. In [10] it is shown that D is closed if and only if the constraints are holonomic. Furthermore, if the kinetic energy is defined by a positive definite metric on Q it can be shown that the implicit generalized Hamiltonian system (T^*Q, D, H) satisfies Assumption 4 [23].

3. Symmetries and first integrals

In this section we investigate the notion of symmetry for implicit generalized Hamiltonian systems. We will recall some important results obtained in [23] and derive some new results. First, we recall some mathematical notation that we will use extensively. This can be found e.g. in Abraham, Marsden and Ratiu [2], Chapters 4 and 6. In the following all manifolds, maps, vector fields and k -forms are assumed to be smooth. Consider two manifolds M and N and a diffeomorphism $\phi : M \rightarrow N$. The push-forward, denoted by ϕ_* , maps a vector field X on M to a vector field $\bar{X} = \phi_* X = T\phi \circ X \circ \phi^{-1}$ on N , where $T\phi$ denotes the tangent of the map ϕ . Instead we define two vector fields X on M and \bar{X} on N to be ϕ -related, denoted by $X \sim_\phi \bar{X}$, if $T\phi \circ X = \bar{X} \circ \phi$ (ϕ not necessarily a diffeomorphism). Recall that if $X \sim_\phi \bar{X}$ and $Y \sim_\phi \bar{Y}$, then $[X, Y] \sim_\phi [\bar{X}, \bar{Y}]$. Let β be a k -form on N . The pull-back, denoted by ϕ^* , maps the k -form β to a k -form $\alpha = \phi^* \beta = T^* \phi \circ \beta \circ \phi$ on M . In local coordinates this reads $(\phi^* \beta)_x(v_1, \dots, v_k) = \beta_{\phi(x)}(T_x \phi \cdot v_1, \dots, T_x \phi \cdot v_k)$, where $v_1, \dots, v_k \in T_x M$. For the special case of a 0-form on N , i.e. a function $F : N \rightarrow \mathbb{R}$, the pull-back is defined as $\phi^* F = F \circ \phi$, which is a function on M .

Now we will turn our attention to symmetries and first integrals of implicit generalized Hamiltonian systems. The notion of symmetry of a generalized Dirac structure was defined in [11].

DEFINITION 5. A vector field $f \in T\mathcal{X}$ is an (*infinitesimal*) *symmetry* of a generalized Dirac structure D on \mathcal{X} if $(L_f X, L_f \alpha) \in D$ for all $(X, \alpha) \in D$.

Analogously, a diffeomorphism $\phi : \mathcal{X} \rightarrow \mathcal{X}$ is called a *symmetry* of D if

$$(\phi_* X, (\phi^*)^{-1} \alpha) \in D \quad (10)$$

for all $(X, \alpha) \in D$ [23].

EXAMPLE 6. Consider the generalized Dirac structure given in Example 1. Then $f \in T\mathcal{X}$ is a symmetry of D if and only if $L_f \omega = 0$.

EXAMPLE 7. Consider the generalized Dirac structure given in Example 2. Then $f \in T\mathcal{X}$ is a symmetry of D if and only if f is canonical with respect to the

Poisson bracket $\{\cdot, \cdot\}$, i.e.

$$L_f\{H_1, H_2\} = \{L_f H_1, H_2\} + \{H_1, L_f H_2\},$$

for all $H_1, H_2 \in C^\infty(\mathcal{X})$.

EXAMPLE 8 [23]. Consider the generalized Dirac structure given in Example 5. Let f be a vector field on T^*Q satisfying $L_f \omega = 0$ and $L_f P_0 \subset P_0$, then f is a symmetry of D .

The following proposition immediately follows from the definition.

PROPOSITION 6 [23]. Let f be a symmetry of a generalized Dirac structure D , then $L_f G_i \subset G_i$, $L_f P_i \subset P_i$, $i = 0, 1$.

The next proposition gives necessary and sufficient conditions for a vector field f to be a symmetry of a generalized Dirac structure D .

PROPOSITION 7. If the vector field f is a symmetry of a generalized Dirac structure D , then

- f is canonical with respect to $\{\cdot, \cdot\}_D$, i.e.

$$L_f\{H_1, H_2\}_D = \{L_f H_1, H_2\}_D + \{H_1, L_f H_2\}_D, \quad \forall H_1, H_2 \in \mathcal{A}_D,$$

- $L_f G_i \subset G_i$, $L_f P_i \subset P_i$, $i = 0, 1$.

If P_1 is constant dimensional and involutive then the converse is also true.

Proof: Take arbitrary $(X_i, dH_i) \in D$, $i = 1, 2$. Because f is a symmetry also $(L_f X_i, L_f dH_i) = (L_f X_i, dL_f H_i) \in D$, $i = 1, 2$. Now,

$$\begin{aligned} L_f\{H_1, H_2\}_D &= L_f\langle dH_1, X_2 \rangle = \langle L_f dH_1, X_2 \rangle + \langle dH_1, L_f X_2 \rangle \\ &= \{L_f H_1, H_2\}_D + \{H_1, L_f H_2\}_D. \end{aligned}$$

Now, suppose P_1 is constant dimensional and involutive. Then ([18], p. 66) $P_1 = \text{span}\{d\beta_i\}$, $\beta_i \in C^\infty(\mathcal{X})$. First we prove that

$$\text{if } (X, dH) \in D, \text{ then } (L_f X, L_f dH) \in D, \quad \forall H \in \mathcal{A}_D. \quad (11)$$

Take arbitrary $H_1, H_2 \in \mathcal{A}_D$, i.e. $(X_i, dH_i) \in D$, $i = 1, 2$. Since

$$\begin{aligned} L_f\{H_1, H_2\}_D &= L_f\langle dH_1, X_2 \rangle = \langle dL_f H_1, X_2 \rangle + \langle dH_1, L_f X_2 \rangle \\ &= \{L_f H_1, H_2\}_D + \langle dH_1, L_f X_2 \rangle \end{aligned}$$

and

$$\{L_f H_1, H_2\}_D + \{H_1, L_f H_2\}_D = \{L_f H_1, H_2\}_D + \langle dH_1, X_{L_f H_2} \rangle$$

(because $L_f P_1 \subset P_1$ we have $L_f dH_2 = dL_f H_2 \in P_1$, i.e., $(X_{L_f H_2}, dL_f H_2) \in D$) it follows from f being canonical that $\langle dH_1, X_{L_f H_2} - L_f X_2 \rangle = 0$, for arbitrary $dH_1 \in P_1$. Because P_1 is spanned by exact one-forms it follows that $X_{L_f H_2} = L_f X_2 + Z$

with $Z \in \ker P_1 = G_0$. Now $(X_{L_f H_2}, L_f dH_2) = (L_f X_2 + Z, L_f dH_2) \in D$ and $Z \in G_0$, i.e. $(Z, 0) \in D$, imply $(L_f X_2, L_f dH_2) \in D$. Since H_2 was arbitrary we have proved (11). Now, because P_1 is spanned by exact one-forms from (11) it follows easily that $(X, \alpha) \in D$ implies $(L_f X, L_f \alpha) \in D$ and so f is a symmetry of D . \square

Another version of Proposition 7 is the following. Define $\{\alpha_1, \alpha_2\} = \langle \alpha_1, X_2 \rangle = -\langle \alpha_2, X_1 \rangle$ for $\alpha_1, \alpha_2 \in P_1$, i.e. $(X_i, \alpha_i) \in D, i = 1, 2$. Then we have the following result

PROPOSITION 8. *f is a symmetry of D if and only if*

- *f is canonical with respect to $\{\cdot, \cdot\}$, i.e.*

$$L_f \{\alpha_1, \alpha_2\} = \{L_f \alpha_1, \alpha_2\} + \{\alpha_1, L_f \alpha_2\}, \quad \forall \alpha_1, \alpha_2 \in P_1;$$

- $L_f G_i \subset G_i, L_f P_i \subset P_i, i = 0, 1$

Proof: Analogously to the proof of Proposition 7. \square

The following proposition says that the set of symmetries of D forms a Lie algebra.

PROPOSITION 9. *Let f_1 and f_2 both be symmetries of a generalized Dirac structure D . Then the Lie bracket $[f_1, f_2]$ is also a symmetry of D .*

Proof: We have

$$L_{[f_1, f_2]} X = [[f_1, f_2], X] = [[f_1, X], f_2] - [[f_2, X], f_1] = L_{f_1} L_{f_2} X - L_{f_2} L_{f_1} X,$$

and $L_{[f_1, f_2]} \alpha = L_{f_1} L_{f_2} \alpha - L_{f_2} L_{f_1} \alpha$, see [2], and the result immediately follows from Definition 5. \square

Now we will turn to the notion of symmetries, and correspondingly first integrals, of implicit (generalized) Hamiltonian systems.

DEFINITION 6. Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) , with D a generalized Dirac structure on \mathcal{X} . We call a nontrivial function $P \in C^\infty(\mathcal{X})$ a *first integral* for (\mathcal{X}, D, H) if

$$\frac{dP}{dt}(x(t)) = \langle dP(x(t)), X_H(x(t)) \rangle = 0, \quad \forall t \in I, \quad (12)$$

for all solutions $x(t)$ of (\mathcal{X}, D, H) , i.e. with $X_H(x(t)) = \dot{x}(t)$.

REMARK 1. Condition (12) can be difficult to check in practice. A sufficient condition for (12) to hold is that

$$\langle dP(x), X_H(x) + G_0(x) \rangle = 0, \quad \forall x \in \mathcal{X}_c,$$

where $X_H(x)$ is arbitrary such that $(X_H(x), dH(x)) \in D(x)$, for every $x \in \mathcal{X}_c$.

We recall the following two results.

PROPOSITION 10 [23, 11, 9]. *Let D be a closed Dirac structure on \mathcal{X} and let $f \in T\mathcal{X}$ for which there exists an $F \in C^\infty(\mathcal{X})$ such that $(f, dF) \in D$. Then f is a symmetry of D .*

PROPOSITION 11 [23]. *Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) . Let $f \in T\mathcal{X}$ for which there exists an $F \in C^\infty(\mathcal{X})$ such that $(f(x), dF(x)) \in D(x), \forall x \in \mathcal{X}_c$. Furthermore, let f be a symmetry of H on \mathcal{X}_c , i.e. $L_f H(x) = 0, \forall x \in \mathcal{X}_c$. Then $L_{X_H} F = 0$ on \mathcal{X}_c , that is, F is a first integral.*

The following proposition says that if the generalized Dirac structure D is closed, then the subset of first integrals in \mathcal{A}_D forms a Lie algebra under the Poisson bracket $\{\cdot, \cdot\}_D$.

PROPOSITION 12. *Consider the implicit Hamiltonian system (\mathcal{X}, D, H) , i.e. with closed Dirac structure D . Let $P_1, P_2 \in C^\infty(\mathcal{X})$ be two first integrals such that $P_1, P_2 \in \mathcal{A}_D$. Then $\{P_1, P_2\}_D$ is also a first integral (with $\{P_1, P_2\}_D \in \mathcal{A}_D$).*

Proof: $P_1, P_2 \in \mathcal{A}_D$, so there exist vector fields X_{P_1}, X_{P_2} such that $(X_{P_1}, dP_1), (X_{P_2}, dP_2) \in D$. Because D is closed, it follows from Theorem 2 that $([X_{P_1}, X_{P_2}], d\{P_1, P_2\}_D) \in D$. Now

$$\begin{aligned} \langle d\{P_1, P_2\}_D(x(t)), X_H(x(t)) \rangle &= -\langle dH(x(t)), [X_{P_1}, X_{P_2}](x(t)) \rangle \\ &= -L_{X_{P_1}}(i_{X_{P_2}} dH)(x(t)) + i_{X_{P_2}}(L_{X_{P_1}} dH)(x(t)) \\ &= 0, \end{aligned}$$

for all solutions $x(t)$ of (\mathcal{X}, D, H) , where we used the fact that $D = D^\perp$ and $i_{X_{P_k}} dH(x(t)) = \langle dH(x(t)), X_{P_k}(x(t)) \rangle = 0, k = 1, 2$, because P_1, P_2 are first integrals. Thus, $\{P_1, P_2\}_D$ is also a first integral of (\mathcal{X}, D, H) . \square

DEFINITION 7. We will call a vector field $f \in T\mathcal{X}$ a *symmetry of the implicit generalized Hamiltonian system (\mathcal{X}, D, H)* if f is a symmetry of the generalized Dirac structure D (as in Definition 5) and f is a symmetry of H , i.e. $L_f H(x(t)) = 0$ for all solutions $x(t)$ of (\mathcal{X}, D, H) , that is, f leaves H invariant (along solutions).

Notice again that a sufficient condition for f to be a symmetry of H is that $L_f H(x) = 0, \forall x \in \mathcal{X}_c$.

The following proposition gives conditions under which a first integral of an implicit Hamiltonian system (\mathcal{X}, D, H) , with closed Dirac structure D , gives rise to a symmetry of the system, see e.g. Proposition 6.31 in [19] for the case of explicit Hamiltonian systems.

PROPOSITION 13. *Consider the implicit Hamiltonian system (\mathcal{X}, D, H) and assume that D is closed. Let P be a first integral such that $P \in \mathcal{A}_D$, i.e. there exists a vector field X_P such that $(X_P, dP) \in D$. Then X_P is a symmetry of (\mathcal{X}, D, H) . Furthermore, X_P generates a one-parameter symmetry group of (\mathcal{X}, D, H) , i.e. the flow of X_P .*

Proof: We have $(X_P(x(t)), dP(x(t))), (X_H(x(t)), dH(x(t))) \in D$ for all solutions $x(t)$ of (\mathcal{X}, D, H) . From $D = D^\perp$ it follows that

$$\langle dH(x(t)), X_P(x(t)) \rangle + \langle dP(x(t)), X_H(x(t)) \rangle = 0. \quad (13)$$

Now, because P is a first integral, from (12) it follows that $L_{X_P}H(x(t)) = \langle dH(x(t)), X_P(x(t)) \rangle = 0$ for all solutions $x(t)$ of (\mathcal{X}, D, H) so X_P is a symmetry of H . Because $(X_P, dP) \in D$ it follows from Proposition 10 that X_P is a symmetry of D . Furthermore, from Remark 14 [23] it is evident that the flow $\phi_t^{X_P}$ of X_P generates a one-parameter symmetry group of (\mathcal{X}, D, H) . \square

A very special sub-class of first integrals is given by the so-called Casimir functions.

DEFINITION 8. Consider a generalized Dirac structure D on \mathcal{X} . A nontrivial function $C \in C^\infty(\mathcal{X})$ is called a *Casimir function* if C is a first integral of (\mathcal{X}, D, H) , as in Definition 6, for every $H \in C^\infty(\mathcal{X})$.

PROPOSITION 14. Consider a generalized Dirac structure D on \mathcal{X} and a function $C \in \mathcal{A}_D$, i.e. $(X_C, dC) \in D$. If $X_C \in G_0$, or equivalently $dC \in P_0$, then C is a Casimir function. If P_1 is constant dimensional and involutive, the converse is also true.

Proof: Take arbitrary $H \in C^\infty(\mathcal{X})$. Like in (13) it follows that

$$\langle dH(x(t)), X_C(x(t)) \rangle + \langle dC(x(t)), X_H(x(t)) \rangle = 0 \quad (14)$$

for all solutions $x(t)$ of (\mathcal{X}, D, H) . Suppose $X_C \in G_0 = \ker P_1$, then $\langle dH(x(t)), X_C(x(t)) \rangle = 0$, and from (14) it follows that C is a first integral of (\mathcal{X}, D, H) . Conversely, suppose C is a Casimir function. Because P_1 is constant dimensional and involutive, there exist local coordinates $(y, s) = (y_1, \dots, y_{n-m}, s_1, \dots, s_m)$ for \mathcal{X} in which $P_1 = \text{span}\{dy_1, \dots, dy_{n-m}\}$. C Casimir means that $\langle dC(x(t)), X_H(x(t)) \rangle = 0$, for all solutions $x(t)$ of (\mathcal{X}, D, H) , for arbitrary $H \in C^\infty(\mathcal{X})$. Take $H_i(y, s) = y_i, i = 1, \dots, n - m$, then $(\mathcal{X}_C)_{H=y_i} = \mathcal{X}$ because $H_i = y_i \in \mathcal{A}_D$, which implies that through each $x \in \mathcal{X}$ there goes a solution $x(t)$ of (\mathcal{X}, D, y_i) . It follows from (14) that $\langle dy_i, X_C \rangle = 0, i = 1, \dots, n - m$, which implies that $X_C \in \ker P_1 = G_0$. \square

Combining Propositions 11 and 13, we get the following Noether type of correspondence between symmetries and first integrals. Note that D being closed implies that the codistribution P_1 is involutive (since by Theorem 2 G_0 is involutive).

PROPOSITION 15. Consider the implicit Hamiltonian system (\mathcal{X}, D, H) and assume that D is closed. If $P \in \mathcal{A}_D$ is a first integral then the corresponding vector field X_P is a symmetry of (\mathcal{X}, D, H) . Conversely, if $X_P \in T\mathcal{X}$ is a symmetry of (\mathcal{X}, D, H) such that $(X_P, dP) \in D$ for some $P \in C^\infty(\mathcal{X})$, then P is a first integral. $\tilde{P} \in C^\infty(\mathcal{X})$ is a second function such that $(X_P, d\tilde{P}) \in D$ only if $\tilde{P} = P + C$ for

some Casimir function $C \in \mathcal{A}_D$. If P_1 is constant dimensional then the converse is also true.

Proof: The first two statements are proved in Propositions 13 and 11, respectively. Now suppose $(X_P, dP), (X_P, d\tilde{P}) \in D$, then it follows that $(0, d(\tilde{P} - P)) \in D$ or $d(\tilde{P} - P) \in P_0$. Proposition 14 implies that $\tilde{P} - P = C$ is a Casimir function. Conversely, suppose that $\tilde{P} - P = C \in \mathcal{A}_D$ is a Casimir function, then $(0, d(\tilde{P} - P)) \in D$. Because also $(X_P, dP) \in D$ it follows that $(X_P, d\tilde{P})$ is also in D . \square

REMARK 2. In this section we derived some results on symmetries and first integrals of Dirac structures and implicit generalized Hamiltonian systems. For some converse results we assumed the constant dimensionality and involutivity of P_1 . We want to remark that in the case of mechanical systems with kinematic constraints, Example 5, the codistribution P_1 is always constant dimensional and involutive.

4. Reduction

In this section we derive some results on the reduction of generalized Dirac structures and correspondingly implicit generalized Hamiltonian systems.

4.1. Reduction of Dirac structures

Investigating reduction of implicit Hamiltonian systems we begin by looking at reduction of Dirac structures. Consider a manifold \mathcal{X} and a generalized Dirac structure D on \mathcal{X} . Let $\tilde{\mathcal{X}}$ be a submanifold of \mathcal{X} , then D induces a generalized Dirac structure \tilde{D} on $\tilde{\mathcal{X}}$. This can be seen by the following. Assume that the distribution G_1 , corresponding to D , is constant dimensional, then by Theorem 3(a) there exists a skew-symmetric linear map $\omega(x) : G_1(x) \rightarrow G_1(x)^*$ such that the generalized Dirac structure D can be written as

$$D = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \alpha(x) - \omega(x)X(x) \in \text{ann } G_1(x), \forall x \in \mathcal{X}, X \in G_1\}. \quad (15)$$

The reduced generalized Dirac structure \tilde{D} on $\tilde{\mathcal{X}}$ is now defined by restricting the map $\omega(x)$ to $G_1(\bar{x}) \cap T_{\bar{x}}\tilde{\mathcal{X}}$, $\bar{x} \in \tilde{\mathcal{X}}$, giving the map $\bar{\omega}(\bar{x})$, i.e.

$$\begin{aligned} \tilde{D} = \{(\bar{X}, \bar{\alpha}) \in T\tilde{\mathcal{X}} \oplus T^*\tilde{\mathcal{X}} \mid \bar{\alpha}(\bar{x}) - \bar{\omega}(\bar{x})\bar{X}(\bar{x}) \in \text{ann } (G_1(\bar{x}) \cap T_{\bar{x}}\tilde{\mathcal{X}}), \\ \bar{X}(\bar{x}) \in G_1(\bar{x}) \cap T_{\bar{x}}\tilde{\mathcal{X}}, \forall \bar{x} \in \tilde{\mathcal{X}}\}, \end{aligned} \quad (16)$$

see also [9]. It follows from Theorem 3(a) (assuming that $G_1(\bar{x}) \cap T_{\bar{x}}\tilde{\mathcal{X}}$ is constant dimensional) that \tilde{D} is a generalized Dirac structure on $\tilde{\mathcal{X}}$. We will show that \tilde{D} can also be written in terms of the inclusion map $\iota : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$.

PROPOSITION 16. *Consider a manifold \mathcal{X} and a generalized Dirac structure D on \mathcal{X} with G_1 constant dimensional. Let $\tilde{\mathcal{X}}$ be a submanifold of \mathcal{X} , and assume that $G_1(\bar{x}) \cap T_{\bar{x}}\tilde{\mathcal{X}}$, $\bar{x} \in \tilde{\mathcal{X}}$, is constant dimensional (on $\tilde{\mathcal{X}}$). Then D induces a generalized*

Dirac structure \bar{D} on $\bar{\mathcal{X}}$ given by

$$\bar{D} = \{(\bar{X}, \bar{\alpha}) \in T\bar{\mathcal{X}} \oplus T^*\bar{\mathcal{X}} \mid \exists X \text{ such that } \bar{X} \sim_i X \text{ and } \exists \alpha \text{ such that } \bar{\alpha} = \iota^* \alpha \text{ with } (X, \alpha) \in D\}. \quad (17)$$

Furthermore, if D is closed then also \bar{D} is closed.

Proof: Denote \bar{D} in (16) by \bar{D}_1 and \bar{D} in (17) by \bar{D}_2 . We prove that $\bar{D}_1 = \bar{D}_2$. $\bar{D}_2 \subset \bar{D}_1$: Let $(\bar{X}, \bar{\alpha}) \in \bar{D}_2$. There exists a vector field $X \in G_1$ such that $\bar{X} \sim_i X$. This means that at points of $\bar{\mathcal{X}}$, X is tangent to $\bar{\mathcal{X}}$, so $\bar{X}(\bar{x}) = X(\bar{x}) \in G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}$ for all $\bar{x} \in \bar{\mathcal{X}}$. Let $\bar{\alpha} = \iota^* \alpha$ where $\alpha(x) - \omega(x)X(x) \in \text{ann } G_1(x)$, $\forall x \in \mathcal{X}$, i.e. $(X, \alpha) \in D$, then $(\iota^* \alpha)(\bar{x}) - \iota^*(\omega X)(\bar{x}) \in \iota^*(\text{ann } G_1)(\bar{x})$, $\forall \bar{x} \in \bar{\mathcal{X}}$, and so, because $\bar{X} \sim_i X$,

$$\bar{\alpha}(\bar{x}) - \bar{\omega}(\bar{x})\bar{X}(\bar{x}) \in \iota^*(\text{ann } G_1)(\bar{x}), \quad \forall \bar{x} \in \bar{\mathcal{X}}.$$

Now, because $\iota^*(\text{ann } G_1)(\bar{x}) \subset \text{ann } (G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}})$, $\forall \bar{x} \in \bar{\mathcal{X}}$, we get

$$\bar{\alpha}(\bar{x}) - \bar{\omega}(\bar{x})\bar{X}(\bar{x}) \in \text{ann } (G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}), \quad \forall \bar{x} \in \bar{\mathcal{X}},$$

which means that $(\bar{X}, \bar{\alpha}) \in \bar{D}_1$.

$\bar{D}_1 \subset \bar{D}_2$: Let $(\bar{X}, \bar{\alpha}) \in \bar{D}_1$. Then $\bar{X}(\bar{x}) \in G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}$, $\forall \bar{x} \in \bar{\mathcal{X}}$. Because G_1 is a smooth subbundle of $T\mathcal{X}$ it follows that \bar{X} can be extended to a vector field $X \in G_1$ such that $\bar{X} \sim_i X$ (one can use the smooth Tietze extension theorem ([2], Theorem 5.5.9), note that X is not unique). There exists an α such that $(X, \alpha) \in D$, i.e. $\alpha(x) - \omega(x)X(x) \in \text{ann } G_1(x)$, $\forall x \in \mathcal{X}$. Then, by the above,

$$(\iota^* \alpha)(x) - \bar{\omega}(\bar{x})\bar{X}(\bar{x}) \in \text{ann } (G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}), \quad \forall \bar{x} \in \bar{\mathcal{X}}, \quad (18)$$

and so, by (16) and (18),

$$\bar{\alpha}(\bar{x}) - (\iota^* \alpha)(\bar{x}) \in \text{ann } (G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}) \subset T_{\bar{x}}^* \bar{\mathcal{X}}, \quad \forall \bar{x} \in \bar{\mathcal{X}}, \quad (19)$$

(note that the annihilation should be taken with respect to $T^*\bar{\mathcal{X}}$). However, because $\bar{\mathcal{X}}$ is a submanifold of \mathcal{X} there exists (locally) a smooth function $F: \mathcal{X} \rightarrow \mathbb{R}^k$, with $k = \text{codim } \bar{\mathcal{X}}$, such that $\bar{\mathcal{X}} = F^{-1}(0)$, i.e. a level set of F . $G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}$ consists of all vector fields $X(\bar{x}) \in G_1(\bar{x})$ which are tangent to $\bar{\mathcal{X}}$, so when we take the annihilator with respect to $T^*\mathcal{X}$,

$$T_{\bar{x}}^* \bar{\mathcal{X}} \supset \text{ann } (G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}}) = \text{span}_{C^\infty(\mathcal{X})} \{dF\}(\bar{x}) + P_0(\bar{x}), \quad \forall \bar{x} \in \bar{\mathcal{X}}.$$

Considered as an element of $T_{\bar{x}}^* \bar{\mathcal{X}}$, that is taking the annihilation with respect to $T^*\bar{\mathcal{X}}$, $dF(\bar{x})$ will be zero, i.e. $\iota^* dF(\bar{x}) = d\iota^* F(\bar{x}) = d0 = 0$, $\forall \bar{x} \in \bar{\mathcal{X}}$. Furthermore, the elements of P_0 will restrict to elements of $\iota^* P_0 \subset T^*\bar{\mathcal{X}}$. Now (19) becomes

$$\bar{\alpha}(\bar{x}) - (\iota^* \alpha)(\bar{x}) \in \iota^* P_0(\bar{x}), \quad \forall \bar{x} \in \bar{\mathcal{X}}.$$

This means that $\bar{\alpha} = \iota^* \alpha + \iota^* \alpha_0$ for some $\alpha_0 \in P_0$. Define $\beta = \alpha + \alpha_0$ then $\bar{\alpha} = \iota^* \beta$ and $(X, \beta) \in D$ (because $(X, \alpha) \in D$ and $(0, \alpha_0) \in D$). Therefore $(\bar{X}, \bar{\alpha}) \in \bar{D}_2$.

Now, assume that D is closed. Take arbitrary $(\bar{X}_k, \bar{\alpha}_k) \in \bar{D}$, $k = 1, 2, 3$, then $\bar{X}_k \sim_i X_k$ and $\bar{\alpha}_k = \iota^* \alpha_k$, with $(X_k, \alpha_k) \in D$ for some X_k and α_k , $k = 1, 2, 3$. Then,

$$\begin{aligned} \langle L_{\bar{X}_1} \bar{\alpha}_2, \bar{X}_3 \rangle + \langle L_{\bar{X}_2} \bar{\alpha}_3, \bar{X}_1 \rangle + \langle L_{\bar{X}_3} \bar{\alpha}_1, \bar{X}_2 \rangle \\ = \langle \iota^* L_{X_1} \alpha_2, \bar{X}_3 \rangle + \langle \iota^* L_{X_2} \alpha_3, \bar{X}_1 \rangle + \langle \iota^* L_{X_3} \alpha_1, \bar{X}_2 \rangle \\ = \langle L_{X_1} \alpha_2, X_3 \rangle + \langle L_{X_2} \alpha_3, X_1 \rangle + \langle L_{X_3} \alpha_1, X_2 \rangle = 0, \end{aligned}$$

because D is closed. This shows that also \bar{D} is closed. \square

There is also a direct proof of Proposition 16, without having to involve (15), (16). Let $\bar{\mathcal{X}}$ be a smooth submanifold of \mathcal{X} and assume that $\bar{\mathcal{X}}$ is closed in \mathcal{X} . Define \bar{D} as in (17). Because D is a linear space, that is $(X_i, \alpha_i) \in D$, $i = 1, 2$, implies $(X_1, \alpha_1) + (X_2, \alpha_2) = (X_1 + X_2, \alpha_1 + \alpha_2) \in D$ and $h(X_1, \alpha_1) = (hX_1, h\alpha_1) \in D$, $\forall h \in C^\infty(\mathcal{X})$, it easily follows that this also holds for \bar{D} . Thus, for every point $\bar{x} \in \bar{\mathcal{X}}$, $\bar{D}(\bar{x})$ is a linear subspace of $T_{\bar{x}}\bar{\mathcal{X}} \times T_{\bar{x}}^*\bar{\mathcal{X}}$. We make the assumption that $\dim(D(\bar{x}) \cap E_s(\bar{x})) = d$, $\forall \bar{x} \in \bar{\mathcal{X}}$, for some integer d (i.e. constant), where E_s is defined as the smooth bundle

$$E_s = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \bar{X} \sim_i X \text{ for some } \bar{X} \in T\bar{\mathcal{X}}\}, \quad (20)$$

(the subscript s stands for submanifold). This assumption equals the condition in [9]. Courant [9] calls $\bar{\mathcal{X}}$ under this assumption a *clean* submanifold of \mathcal{X} .

PROPOSITION 17. *Assume that $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in \bar{\mathcal{X}}$, is constant dimensional on $\bar{\mathcal{X}}$. Then \bar{D} defined in (17) is a generalized Dirac structure on $\bar{\mathcal{X}}$.*

Proof: We prove that $\bar{D} = \bar{D}^\perp$. The first inclusion, i.e. $\bar{D} \subset \bar{D}^\perp$, is easy. We prove the second inclusion, i.e. $\bar{D}^\perp \subset \bar{D}$. Take an arbitrary pair $(\bar{Y}, \bar{\beta}) \in \bar{D}^\perp$, that is

$$(\bar{Y}, \bar{\beta}) \in T\bar{\mathcal{X}} \oplus T^*\bar{\mathcal{X}} \text{ such that } \langle \bar{\beta}, \bar{X} \rangle + \langle \bar{\alpha}, \bar{Y} \rangle = 0, \quad \forall (\bar{X}, \bar{\alpha}) \in \bar{D}.$$

There exist $Y \in T\mathcal{X}$ such that $\bar{Y} \sim_i Y$ and $\beta \in T^*\mathcal{X}$ such that $\bar{\beta} = \iota^* \beta$ (because ι^* is surjective). Notice that this only defines Y and β at points $\bar{x} \in \bar{\mathcal{X}} \subset \mathcal{X}$. Now,

$$0 = \langle \bar{\beta}, \bar{X} \rangle + \langle \bar{\alpha}, \bar{Y} \rangle = \langle \iota^* \beta, \bar{X} \rangle + \langle \iota^* \alpha, \bar{Y} \rangle = (\langle \beta, X \rangle + \langle \alpha, Y \rangle) \circ \iota,$$

which means that $\langle \beta, X \rangle(\bar{x}) + \langle \alpha, Y \rangle(\bar{x}) = 0$, for all $\bar{x} \in \bar{\mathcal{X}} \subset \mathcal{X}$ and all pairs $(X, \alpha) \in D$ for which $\bar{X} \sim_i X$ for some $\bar{X} \in T\bar{\mathcal{X}}$. Therefore,

$$\begin{aligned} (Y, \beta)(\bar{x}) &\in [(D \cap E_s)(\bar{x})]^\perp = [D(\bar{x}) \cap E_s(\bar{x})]^\perp \\ &= D(\bar{x}) + [E_s(\bar{x})]^\perp = D(\bar{x}) + (0, \text{ann } T_{\bar{x}}\bar{\mathcal{X}}) \end{aligned} \quad (21)$$

for all $\bar{x} \in \bar{\mathcal{X}} \subset \mathcal{X}$, with E_s defined as in (20) (and where we used the assumption on constant dimensionality at the first equality, see e.g. [12]).

Consider $\tilde{E}_s = \{(0, \gamma) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \iota^*\gamma = 0\}$, then \tilde{E}_s is a smooth bundle. Indeed, $\tilde{E}_s(x)$, $x \notin \tilde{\mathcal{X}}$, can locally (that is, in some neighbourhood $U \subset \mathcal{X}$ of x , $U \cap \tilde{\mathcal{X}} = \emptyset$) be written as

$$\tilde{E}_s(x) = \text{span}_{C^\infty(\mathcal{X})}\{(0, dx_1), \dots, (0, dx_n)\},$$

where x_1, \dots, x_n are local coordinates for \mathcal{X} around x . Consider a point $\bar{x} \in \tilde{\mathcal{X}}$. Because $\tilde{\mathcal{X}}$ is a submanifold of \mathcal{X} , there exist local coordinates $x_1, \dots, x_m, x_{m+1}, \dots, x_n$ for \mathcal{X} in some neighbourhood U of \bar{x} such that x_1, \dots, x_m are local coordinates for $\tilde{\mathcal{X}}$. Then $\tilde{E}_s(x)$ can be written as

$$\tilde{E}_s(x) = \text{span}_{C^\infty(\mathcal{X})}\{f_1(x)(0, dx_1), \dots, f_m(x)(0, dx_m), (0, dx_{m+1}), \dots, (0, dx_n)\},$$

for all $x \in U$, with $f_1, \dots, f_m \in C^\infty(U)$ such that $f_i(x) = 0 \Leftrightarrow x \in \tilde{\mathcal{X}}$.

Notice that $\tilde{E}_s(\bar{x}) = (0, \text{ann } T_{\bar{x}}\tilde{\mathcal{X}})$ for all $\bar{x} \in \tilde{\mathcal{X}}$. Then (21) becomes

$$(Y, \beta)(\bar{x}) \in D(\bar{x}) + \tilde{E}_s(\bar{x}), \quad (22)$$

for all $\bar{x} \in \tilde{\mathcal{X}} \subset \mathcal{X}$. Because D is also a smooth bundle (by definition), around every point $x \in \mathcal{X}$ there exists a local basis $(X_i, \alpha_i) \in D$, $i = 1, \dots, n$ —where X_i and α_i are locally (that is, around x) smooth vector fields, respectively one-forms—such that locally $D = \text{span}_{C^\infty(\mathcal{X})}\{(X_i, \alpha_i)\}$. From (22) it follows that we can write

$$(Y, \beta)(\bar{x}) = \sum_{i=1}^n h_i(\bar{x})(X_i, \alpha_i)(\bar{x}) + \sum_{j=m+1}^n g_j(\bar{x})(0, dx_j) \quad (23)$$

for some functions $h_i, g_j \in C^\infty(U)$, $i = 1, \dots, n$, $j = m+1, \dots, n$, $U \subset \mathcal{X}$ a neighbourhood of \bar{x} . Define

$$\gamma(\bar{x}) = \sum_{j=m+1}^n g_j(\bar{x})dx_j,$$

then from (23) $(Y, \beta - \gamma)(\bar{x}) \in D(\bar{x})$, $\forall \bar{x} \in \tilde{\mathcal{X}} \subset \mathcal{X}$. Because of (23) $(Y, \beta - \gamma)$ can be locally, that is in some neighbourhood $U \subset \mathcal{X}$ of every \bar{x} , extended to a smooth pair (Y_e, β_e) defined on U such that $Y_e(\bar{x}) = Y(\bar{x})$, $\beta_e(\bar{x}) = \beta(\bar{x}) - \gamma(\bar{x})$, $\forall \bar{x} \in U \cap \tilde{\mathcal{X}}$, and $(Y_e, \beta_e)(x) \in D(x)$, $\forall x \in U$. Indeed, take

$$(Y_e, \beta_e)(x) = \sum_{i=1}^n h_i(x)(X_i, \alpha_i)(x), \quad x \in U.$$

Then, by the smooth Tietze extension theorem ([2], Theorem 5.5.9), $(Y, \beta - \gamma)$ can be globally extended to a pair

$$(Y', \beta') \in D \quad (24)$$

such that $Y'(\bar{x}) = Y(\bar{x})$, $\beta'(\bar{x}) = \beta(\bar{x}) - \gamma(\bar{x})$, $\forall \bar{x} \in \bar{\mathcal{X}} \subset \mathcal{X}$. It follows that

$$Y' \sim_i \bar{Y} \quad (25)$$

and

$$\iota^* \beta' = \iota^*(\beta - \gamma) = \iota^* \beta - 0 = \bar{\beta}, \quad (26)$$

where we used that $\iota^* \beta'$ only depends on the definition of β' in the points $\bar{x} \in \bar{\mathcal{X}} \subset \mathcal{X}$. Now Eqs. (24–26) imply that $(\bar{Y}, \bar{\beta}) \in \bar{D}$. So we have proved that $\bar{D}^\perp \subset \bar{D}$. So $\bar{D} = \bar{D}^\perp$. Smoothness of the pairs $(\bar{X}, \bar{\alpha}) \in \bar{D}$ comes from smoothness of D , and thus \bar{D} is a generalized Dirac structure on $\bar{\mathcal{X}}$. \square

REMARK 3. With respect to the comparison of Propositions 16 and 17 we remark that (i) G_1 and $G_1(\bar{x}) \cap T_{\bar{x}} \bar{\mathcal{X}}$, $\bar{x} \in \bar{\mathcal{X}}$, constant dimensional imply $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in \bar{\mathcal{X}}$, constant dimensional, and (ii) G_1 and $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in \bar{\mathcal{X}}$, constant dimensional imply $G_1(\bar{x}) \cap T_{\bar{x}} \bar{\mathcal{X}}$, $\bar{x} \in \bar{\mathcal{X}}$, constant dimensional.

Consider a manifold \mathcal{X} and a generalized Dirac structure D on \mathcal{X} . Consider a symmetry Lie group G of D , that is, every $g \in G$ induces an action $\phi_g : \mathcal{X} \rightarrow \mathcal{X}$ on \mathcal{X} , which is a diffeomorphism, and ϕ_g is a symmetry of the generalized Dirac structure D . Equivalently, let \mathcal{G} be the Lie algebra corresponding to G , then for every $\xi \in \mathcal{G}$ the infinitesimal generator $\xi_{\mathcal{X}}$ is an (infinitesimal) symmetry of D as in Definition 5. Then the generalized Dirac structure D on \mathcal{X} induces a generalized Dirac structure \hat{D} on the quotient space $\hat{\mathcal{X}} = \mathcal{X}/G$ of G -orbits on \mathcal{X} . Throughout we assume that $\hat{\mathcal{X}} = \mathcal{X}/G$ has a manifold structure. The usual assumption made is that G acts freely and properly on \mathcal{X} , see [1]. Furthermore, in Proposition 18 we need the following assumptions. Let V denote the distribution spanned by the infinitesimal generators of G . Assume that $V + G_0$ is constant dimensional. Furthermore, define the smooth bundle

$$E_q = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \alpha = \pi^* \hat{\alpha} \text{ for some } \hat{\alpha} \in T^*\hat{\mathcal{X}}\}, \quad (27)$$

(the subscript q stands for quotient manifold). We assume that $D \cap E_q$ is constant dimensional (on \mathcal{X}). The next proposition was given in [23], an extended proof is given here.

PROPOSITION 18 [23]. *Consider a manifold \mathcal{X} and a generalized Dirac structure D on \mathcal{X} . Let G be a symmetry Lie group of D and assume that $V + G_0$ and $D \cap E_q$ are constant dimensional. Then D induces a generalized Dirac structure \hat{D} on $\hat{\mathcal{X}} = \mathcal{X}/G$ given by*

$$\hat{D} = \{(\hat{X}, \hat{\alpha}) \in T\hat{\mathcal{X}} \oplus T^*\hat{\mathcal{X}} \mid \exists X \text{ such that } X \sim_{\pi} \hat{X} \text{ and } (X, \alpha) \in D \\ \text{where } \alpha = \pi^* \hat{\alpha}\}.$$

Here, $\pi : \mathcal{X} \rightarrow \hat{\mathcal{X}} = \mathcal{X}/G$ is the projection map. Furthermore, if D is closed then also \hat{D} is closed.

Proof: We show that \hat{D} is a generalized Dirac structure. The first inclusion $\hat{D} \subset \hat{D}^\perp$ is easy. We prove the second inclusion, $\hat{D}^\perp \subset \hat{D}$. Take an arbitrary pair $(\hat{Y}, \hat{\beta}) \in \hat{D}^\perp$, that is

$$(\hat{Y}, \hat{\beta}) \in T\hat{\mathcal{X}} \oplus T^*\hat{\mathcal{X}} \text{ such that } \langle \hat{\beta}, \hat{X} \rangle + \langle \hat{\alpha}, \hat{Y} \rangle = 0, \quad \forall (\hat{X}, \hat{\alpha}) \in \hat{D}. \quad (28)$$

Let $Y \in T\mathcal{X}$ be such that $Y \sim_\pi \hat{Y}$ and define $\beta = \pi^*\hat{\beta}$, then (28) becomes

$$0 = (\langle \hat{\beta}, \hat{X} \rangle + \langle \hat{\alpha}, \hat{Y} \rangle) \circ \pi = \langle \beta, X \rangle + \langle \alpha, Y \rangle, \quad (29)$$

for all $(X, \alpha) \in D$ for which $X \sim_\pi \hat{X}$ and $\alpha = \pi^*\hat{\alpha}$ for some $\hat{X} \in T\hat{\mathcal{X}}$, $\hat{\alpha} \in T^*\hat{\mathcal{X}}$. Now consider an arbitrary $(X, \alpha) \in D$ with $\alpha = \pi^*\hat{\alpha}$ for some $\hat{\alpha} \in T^*\hat{\mathcal{X}}$. Since G is a symmetry group, $(L_{\xi_X}X, L_{\xi_X}\pi^*\hat{\alpha}) \in D$ for all infinitesimal generators ξ_X , $\xi \in \mathcal{G}$. Since $L_{\xi_X}\pi^*\hat{\alpha} = 0$, this yields

$$L_{\xi_X}X \in G_0, \quad \forall \xi_X, \xi \in \mathcal{G}. \quad (30)$$

Furthermore, by Proposition 6, $L_{\xi_X}G_0 \subset G_0$. Take an arbitrary $v = \sum_i h_i(\xi_i)_X \in V$, where $\{\xi_i\}_i$ is a basis of \mathcal{G} and $h_i \in C^\infty(\mathcal{X})$, then by (30)

$$[X, v] = \sum_i h_i[X, (\xi_i)_X] + \sum_i L_X h_i (\xi_i)_X \in G_0 + V,$$

so $[X, V] \subset V + G_0$. Analogously, it follows that $[G_0, V] \subset V + G_0$. Now, since $V + G_0$ is constant dimensional we have the following properties (see [12, 18] for the analogue in controlled invariant distributions)

- (a) there exist Z_1, \dots, Z_k which span G_0 such that $[Z_i, V] \subset V$, which implies that $Z_i \sim_\pi \hat{Z}_i$ for some $\hat{Z}_i \in T\hat{\mathcal{X}}$, $i = 1, \dots, k$,
- (b) there exists a $Z \in G_0$ such that $[X+Z, V] \subset V$, which implies that $X+Z \sim_\pi \hat{X}$ for some $\hat{X} \in T\hat{\mathcal{X}}$.

Take an arbitrary $Z \in G_0$ such that $Z \sim_\pi \hat{Z}$ for some $\hat{Z} \in T\hat{\mathcal{X}}$, then by (29) it follows that $\langle \pi^*\hat{\beta}, Z \rangle = 0$. Therefore $\langle \pi^*\hat{\beta}, Z_i \rangle = 0$, $i = 1, \dots, k$, and since Z_1, \dots, Z_k span G_0 ,

$$\langle \pi^*\hat{\beta}, G_0 \rangle = 0. \quad (31)$$

Now take any pair $(X, \alpha) \in D$ for which there exists an $\hat{\alpha} \in T^*\hat{\mathcal{X}}$ such that $\alpha = \pi^*\hat{\alpha}$. Then by (b) there exists a $Z \in G_0$ (so $(X+Z, \alpha) \in D$) such that $X+Z \sim_\pi \hat{X}$ for some $\hat{X} \in T\hat{\mathcal{X}}$, and so by (29) $\langle \beta, X+Z \rangle + \langle \alpha, Y \rangle = 0$, which by (31) and the fact that $\beta = \pi^*\hat{\beta}$ implies

$$\langle \beta, X \rangle + \langle \alpha, Y \rangle = 0. \quad (32)$$

Thus we have shown that (29), or (32), holds for all $(X, \alpha) \in D$ such that $\alpha = \pi^*\hat{\alpha}$ for some $\hat{\alpha} \in T^*\hat{\mathcal{X}}$. Hence

$$(Y, \beta) \in (D \cap E_q)^\perp = D + E_q^\perp, \quad (33)$$

where we used the constant dimensionality of $D \cap E_q$. We claim that

$$E_q^\perp = \{(\tilde{X}, 0) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \tilde{X} \sim_\pi 0\}. \quad (34)$$

Indeed, the inclusion \supset is obvious, while for the reverse inclusion note that if $(\tilde{X}, \tilde{\alpha})$ is such that $\langle \tilde{\alpha}, X \rangle + \langle \alpha, \tilde{X} \rangle = 0$, for all $(X, \alpha) \in E_q$, then (taking $X = 0$) $\langle \alpha, \tilde{X} \rangle = 0$ for all $\alpha = \pi^*\hat{\alpha}$, $\hat{\alpha} \in T^*\hat{\mathcal{X}}$, and thus $\tilde{X} \sim_\pi 0$. Hence $0 = \langle \tilde{\alpha}, X \rangle + \langle \alpha, \tilde{X} \rangle = \langle \tilde{\alpha}, X \rangle$, for all $X \in T\mathcal{X}$, implying that $\tilde{\alpha} = 0$. This proves the claim. By Eqs. (33), (34) there exists a vector field $\tilde{Y} \in T\mathcal{X}$, with $\tilde{Y} \sim_\pi 0$, such that $(Y + \tilde{Y}, \beta) \in D$. Since $Y + \tilde{Y} \sim_\pi \hat{Y}$ this implies that $(\hat{Y}, \hat{\beta}) \in \hat{D}$. This shows that $\hat{D}^\perp \subset \hat{D}$. So $\hat{D} = \hat{D}^\perp$, which means that \hat{D} is a generalized Dirac structure on $\hat{\mathcal{X}}$. For the proof that the closedness of D implies the closedness of \hat{D} we refer to [23]. \square

REMARK 4. Take $\hat{H}_1, \hat{H}_2 \in \mathcal{A}_{\hat{D}}$, i.e. $(\hat{X}_1, d\hat{H}_1), (\hat{X}_2, d\hat{H}_2) \in \hat{D}$. Then $(X_1, dH_1), (X_2, dH_2) \in D$ for $X_j \sim_\pi \hat{X}_j$ and $H_j = \hat{H}_j \circ \pi$, $j = 1, 2$. So the bracket of admissible functions becomes

$$\{\hat{H}_1, \hat{H}_2\}_{\hat{D}}(\hat{x}) = \langle d\hat{H}_2, \hat{X}_1 \rangle(\hat{x}) = \langle dH_2, X_1 \rangle(x) = \{H_1, H_2\}_D(x),$$

where $\pi(x) = \hat{x}$. Equivalently

$$\{\hat{H}_1, \hat{H}_2\}_{\hat{D}} \circ \pi = \{\hat{H}_1 \circ \pi, \hat{H}_2 \circ \pi\}_D. \quad (35)$$

4.2. Reduction of implicit generalized Hamiltonian systems

In this section we will investigate the reduction possibilities of implicit generalized Hamiltonian systems. We begin by stating the analogies of Propositions 17 and 18.

Consider an implicit generalized Hamiltonian system (\mathcal{X}, D, H) . Let $P \in C^\infty(\mathcal{X})$ be a first integral of (\mathcal{X}, D, H) as in Definition 6, and consider the level set $\tilde{\mathcal{X}} = \{x \in \mathcal{X} \mid P(x) = a\}$ for some $a \in \mathbb{R}$ such that $\tilde{\mathcal{X}} \cap \mathcal{X}_c$ is nonempty. Then every solution of (\mathcal{X}, D, H) starting in $\tilde{\mathcal{X}}$ will remain in $\tilde{\mathcal{X}}$. We can describe these solutions by using the induced Dirac structure on $\tilde{\mathcal{X}}$.

PROPOSITION 19. *Consider the assumptions described above. Let $D(\tilde{x}) \cap E_s(\tilde{x})$, $\tilde{x} \in \tilde{\mathcal{X}}$, be constant dimensional on $\tilde{\mathcal{X}}$, where E_s is defined in (20). Then every solution of (\mathcal{X}, D, H) lying in $\tilde{\mathcal{X}}$ is a solution of the implicit generalized Hamiltonian system $(\tilde{\mathcal{X}}, \tilde{D}, \tilde{H})$, where \tilde{D} is the generalized Dirac structure induced by D , see Proposition 17, and $\tilde{H} = \iota^*H$, i.e. the Hamiltonian H restricted to $\tilde{\mathcal{X}}$.*

Proof: Let $x(t)$ be a solution of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) contained in $\tilde{\mathcal{X}}$, i.e. $(X_H, dH)(x(t)) \in D(x(t))$, for all $t \in I$, where $X_H(x(t)) = \dot{x}(t)$ and $I \subset \mathbb{R}$ is the interval of existence of $x(t)$. Because P is a

first integral it follows that $X_H(x(t))$ is tangent to $\bar{\mathcal{X}}$ at all times t , see also (12). Define $X_{\bar{H}}$ such that

$$T_{\bar{x}(t)}\iota \cdot X_{\bar{H}}(\bar{x}(t)) = X_H(x(t)), \quad \forall t \in I, \quad (36)$$

where $\iota(\bar{x}(t)) = x(t)$. Take arbitrary $(\bar{Y}, \bar{\beta}) \in \bar{D}$. There exist $(Y, \beta) \in D$ such that $\bar{Y} \sim_\iota Y$ and $\bar{\beta} = \iota^* \beta$. Then

$$\begin{aligned} (\langle d\bar{H}, \bar{Y} \rangle + \langle \bar{\beta}, X_{\bar{H}} \rangle)(\bar{x}(t)) &= (\langle \iota^* dH, \bar{Y} \rangle + \langle \iota^* \beta, X_{\bar{H}} \rangle)(\bar{x}(t)) \\ &= (\langle dH, Y \rangle + \langle \beta, X_H \rangle)(x(t)) = 0, \end{aligned}$$

where in the last step we used that $(X_H, dH)(x(t)) \in D(x(t)) = D^\perp(x(t)) = [D(x(t))]^\perp$, $\forall t \in I$, by Proposition 1. This shows that

$$(X_{\bar{H}}, d\bar{H})(\bar{x}(t)) \in [\bar{D}(\bar{x}(t))]^\perp = \bar{D}^\perp(\bar{x}(t)) = \bar{D}(\bar{x}(t)), \quad \forall t \in I,$$

which implies that $\bar{x}(t)$ is a solution of $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$. (Note that by (36) $\dot{\bar{x}}(t) = X_{\bar{H}}(\bar{x}(t))$.) \square

REMARK 5. This proposition can be easily extended to the case where we consider the level set $\bar{\mathcal{X}} = \{x \in \mathcal{X} \mid P_1(x) = a_1, \dots, P_r(x) = a_r, (a_1, \dots, a_r) \in \mathbb{R}^r\}$ of r independent first integrals $P_1, \dots, P_r \in C^\infty(\mathcal{X})$ of (\mathcal{X}, D, H) . Furthermore, it is clear that Proposition 19 can also be extended to the case of an arbitrary submanifold $\bar{\mathcal{X}}$ of \mathcal{X} left invariant by the Hamiltonian flow (i.e. X_H is tangent to $\bar{\mathcal{X}}$).

Proposition 19 says that every solution of (\mathcal{X}, D, H) lying in $\bar{\mathcal{X}}$ is a solution of $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$. However, in general, $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$ will generate more solutions, i.e. solutions that do not correspond to any solution of (\mathcal{X}, D, H) . This can be seen most easily in the classical case of reduction of a Hamiltonian system on a symplectic manifold N to a submanifold M of N . In general, the Hamiltonian system will reduce to a Hamiltonian system on a *presymplectic* submanifold M , meaning that the induced 2-form $\bar{\omega}$ has nontrivial kernel. Due to this nontriviality the reduced system will generate certain solutions not corresponding to solutions of the original system.

An example where the above cannot happen is when we restrict a Hamiltonian system on a Poisson manifold to a level set of a Casimir function. Then the solutions of the restricted system will all correspond to solutions of the original system. More generally we can say the following.

PROPOSITION 20. Consider an implicit generalized Hamiltonian system (\mathcal{X}, D, H) . Let $C \in C^\infty(\mathcal{X})$ be a Casimir function of (\mathcal{X}, D, H) , as in Definition 8, and assume that $dC \in \mathcal{P}_0$. Consider the level set $\bar{\mathcal{X}} = \{x \in \mathcal{X} \mid C(x) = a\}$ for some $a \in \mathbb{R}$ such that $\bar{\mathcal{X}} \cap \mathcal{X}_c$ is nonempty. Then the solutions of (\mathcal{X}, D, H) lying in $\bar{\mathcal{X}}$ are exactly the solutions of the implicit generalized Hamiltonian system $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$, where \bar{D} is the generalized Dirac structure induced by D , see Proposition 17, and $\bar{H} = \iota^* H$, i.e. the Hamiltonian H restricted to $\bar{\mathcal{X}}$.

Proof: First note that since $dC \in P_0 = \text{ann } G_1$, $D(\bar{x}) \cap E_s(\bar{x}) = D(\bar{x})$, $\bar{x} \in \bar{\mathcal{X}}$, is constant dimensional on $\bar{\mathcal{X}}$. See the proof of Proposition 19 to conclude that every solution $x(t)$ of (\mathcal{X}, D, H) is a solution of $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$.

Now, let $\bar{x}(t)$ be a solution of $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$, i.e. $(X_{\bar{H}}, d\bar{H})(\bar{x}(t)) \in \bar{D}(\bar{x}(t))$, for all $t \in I$, where $X_{\bar{H}}(\bar{x}(t)) = \dot{\bar{x}}(t)$. Define

$$X_H(x(t)) = T_{\bar{x}(t)}\iota \cdot X_{\bar{H}}(\bar{x}(t)), \quad \forall t \in I, \quad (37)$$

where $x(t) = \iota(\bar{x}(t))$. Take arbitrary $(Y, \beta) \in D$. Because $dC \in P_0$ it follows that $\langle dC, Y \rangle(x) = 0$, $\forall x \in \mathcal{X}$. This means that Y is tangent to $\bar{\mathcal{X}}$. Define $\bar{Y} \in T\bar{\mathcal{X}}$ such that $T_{\bar{x}}\iota \cdot \bar{Y}(\bar{x}) = Y(\iota(\bar{x}))$, $\forall \bar{x} \in \bar{\mathcal{X}}$, and $\bar{\beta} = \iota^*\beta$. Then

$$\begin{aligned} (\langle dH, Y \rangle + \langle \beta, X_H \rangle)(x(t)) &= (\langle \iota^*dH, \bar{Y} \rangle + \langle \iota^*\beta, X_{\bar{H}} \rangle)(\bar{x}(t)) \\ &= (\langle d\bar{H}, \bar{Y} \rangle + \langle \bar{\beta}, X_{\bar{H}} \rangle)(\bar{x}(t)) = 0, \end{aligned}$$

where in the last step we used that $(X_{\bar{H}}, d\bar{H})(\bar{x}(t)) \in \bar{D}(\bar{x}(t)) = \bar{D}^\perp(\bar{x}(t)) = [\bar{D}(\bar{x}(t))]^\perp$, $\forall t \in I$. This shows that

$$(X_H, dH)(x(t)) \in [D(x(t))]^\perp = D^\perp(x(t)) = D(x(t)), \quad \forall t \in I.$$

Since $\dot{x}(t) = X_H(x(t))$ by (37), this means that $x(t)$ is a solution of (\mathcal{X}, D, H) . \square

REMARK 6. Of course, this proposition can also be easily extended to the case of multiple independent Casimir functions, or to the case of an arbitrary submanifold $\bar{\mathcal{X}}$ of \mathcal{X} with the property that every $Y \in G_1$ is tangent to $\bar{\mathcal{X}}$ (see also Remark 5).

A nice example of the reduction to submanifolds described above is given by the following.

EXAMPLE 9. Consider an implicit Hamiltonian system (\mathcal{X}, D, H) (i.e. with D being closed), and assume that G_1 is constant dimensional. Then D can be written in the form (2). Since D is closed, G_1 is involutive, Theorem 2. Hence by the Frobenius theorem around every point $x \in \mathcal{X}$ there locally exists a submanifold $\bar{\mathcal{X}}$ of \mathcal{X} such that $\bar{\mathcal{X}}$ is an integral manifold of G_1 , i.e. $T\bar{\mathcal{X}} = G_1(x)$ (actually this defines a foliation of integral manifolds of G_1). Since $D(\bar{x}) \cap E_s(\bar{x}) = D(\bar{x})$, $\bar{x} \in \bar{\mathcal{X}}$, is constant dimensional on $\bar{\mathcal{X}}$ we can use Proposition 20 (see also Remark 6) to reduce the system (\mathcal{X}, D, H) to an implicit generalized Hamiltonian system $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$ on $\bar{\mathcal{X}}$. Since D has the form (2), \bar{D} will be given by

$$\bar{D} = \{(\bar{X}, \bar{\alpha}) \in T\bar{\mathcal{X}} \oplus T^*\bar{\mathcal{X}} \mid \bar{\alpha}(\bar{x}) = \bar{\omega}(\bar{x})\bar{X}(\bar{x}), \quad \forall \bar{x} \in \bar{\mathcal{X}}\}, \quad (38)$$

where $\bar{\omega}$ is the restriction of ω to G_1 . $\bar{\omega} : T\bar{\mathcal{X}} \rightarrow T^*\bar{\mathcal{X}}$ is a closed 2-form on $\bar{\mathcal{X}}$, with kernel G_0 . So \bar{D} in (38) represents a presymplectic structure on $\bar{\mathcal{X}}$. This corresponds to the theorem in Courant [9] stating that a closed Dirac structure has a foliation by presymplectic leaves. Concluding, we see that the implicit Hamiltonian system (\mathcal{X}, D, H) reduces to the presymplectic Hamiltonian system $(\bar{\mathcal{X}}, \bar{D}, \bar{H})$.

Notice that in the case when D represents a Poisson structure, with the Poisson bracket satisfying the Jacobi identity (see Examples 2 and 4), \mathcal{X} is exactly a symplectic submanifold of \mathcal{X} . (This can be seen by using the Darboux theorem, since the bracket satisfies the Jacobi identity and $G_1 = \text{Im } J$ i.e. the rank of the matrix J is constant dimensional.) The (already explicit) system (\mathcal{X}, D, H) , given locally by (6), reduces to the system $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$ given locally by

$$\dot{q} = \frac{\partial \hat{H}}{\partial p}(q, p), \quad \dot{p} = -\frac{\partial \hat{H}}{\partial q}(q, p),$$

where $\hat{H}(q, p) = H(q, p, r_0)$ with $r_0 = r(0)$.

To state the analogue of Proposition 18 we first need the following.

DEFINITION 9. We will call a vector field $f \in T\mathcal{X}$ a *strong symmetry* of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) if f is a symmetry of the generalized Dirac structure D (as in Definition 5) and f leaves H invariant everywhere, i.e. $L_f H(x) = 0$, $\forall x \in \mathcal{X}$ (note the difference with Definition 7). G is called a *strong symmetry Lie group* of (\mathcal{X}, D, H) if G is a symmetry Lie group of D (as in Proposition 18) and every infinitesimal generator $\xi_{\mathcal{X}}$, $\xi \in \mathcal{G}$, leaves H invariant everywhere.

A preliminary version of the next proposition was stated in [23].

PROPOSITION 21. Consider an implicit generalized Hamiltonian system (\mathcal{X}, D, H) . Let G be a strong symmetry Lie group of (\mathcal{X}, D, H) and assume that $V + G_0$ and $D \cap E_q$ are constant dimensional. Then (\mathcal{X}, D, H) projects to the implicit generalized Hamiltonian system $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$, where $\hat{\mathcal{X}} = \mathcal{X}/G$, \hat{D} is the generalized Dirac structure induced by D , see Proposition 18, and the Hamiltonian \hat{H} is such that $H = \hat{H} \circ \pi$ (note that G leaves H invariant so \hat{H} is well defined).

More explicitly: Every solution $\hat{x}(t)$ of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$ is (locally) the projection under π of a solution $x(t)$ of (\mathcal{X}, D, H) . Conversely, let $x(t)$ be a solution of (\mathcal{X}, D, H) along a projectable vector field X_H , that is, assume that there exists a vector field $X \in T\mathcal{X}$ such that $X \sim_{\pi} \hat{X}$ for some $\hat{X} \in T\hat{\mathcal{X}}$ and $X(x(t)) = X_H(x(t))$, $\forall t \in I$, then $x(t)$ can be projected to a solution $\hat{x}(t)$ of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$.

Proof: Let $\hat{x}(t)$ be a solution of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$, i.e. $(X_{\hat{H}}, d\hat{H})(\hat{x}(t)) \in \hat{D}(\hat{x}(t))$, for all $t \in I$, where $X_{\hat{H}}(\hat{x}(t)) = \dot{\hat{x}}(t)$ and $I \subset \mathbb{R}$ is the interval of existence of $\hat{x}(t)$. Define $\hat{A} = \{\hat{x}(t) \mid t \in I\}$, and assume that \hat{A} is a closed subset of $\hat{\mathcal{X}}$. If this is not the case, for instance if $\hat{x}(t)$ converges asymptotically to an equilibrium point, then by defining \hat{A} on any closed interval $I' \subset I$ (i.e. by considering $\hat{x}(t)$ only "locally") \hat{A} can be made into a closed subset of $\hat{\mathcal{X}}$. Then it follows that $(X_{\hat{H}}, d\hat{H})(\hat{x}) \in \hat{D}(\hat{x})$, $\forall \hat{x} \in \hat{A}$. Because \hat{D} is a smooth bundle the pair $(X_{\hat{H}}, d\hat{H})$ can be locally extended to a pair in \hat{D} , and therefore also, by the smooth Tietze

extension theorem [2], globally extended to a pair $(\hat{X}, \hat{\alpha}) \in \hat{D}$. By definition of \hat{D} there exists a pair $(X, \alpha) \in D$ where

$$X \sim_{\rho} \hat{X}, \alpha = \rho^* \hat{\alpha}. \quad (39)$$

Because

$$\rho^* \hat{\alpha}(x) = \hat{\alpha}(\rho(x))(T_x \rho \cdot) = d\hat{H}(\rho(x))(T_x \rho \cdot) = \rho^* d\hat{H}(x) = dH(x),$$

for all $x \in \mathcal{X}$ such that $\rho(x) = \hat{x} \in \hat{A}$, it follows that $(X, dH)(x) \in D(x)$ for all $x \in \mathcal{X}$ such that $\rho(x) = \hat{x} \in \hat{A}$. Equivalently, let $x(t)$ be such that $\dot{x}(t) = X(x(t))$, then $\rho(x(t)) = \hat{x}(t)$ (because of (39)), and $(X_H, dH)(x(t)) \in D(x(t))$, for all $t \in I$, where we wrote X_H for X . This means that $x(t)$ is a solution of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) .

Conversely, let $x(t)$ be a solution of (\mathcal{X}, D, H) , i.e. $(X_H, dH)(x(t)) \in D(x(t))$, for all $t \in I$, where $X_H(x(t)) = \dot{x}(t)$ and $I \subset \mathbb{R}$ is the interval of existence of $x(t)$. Assume that $x(t)$ is the flow of a projectable vector field, that is, assume that there exists a vector field $X \in T\mathcal{X}$ such that $X \sim_{\pi} \hat{X}$ for some $\hat{X} \in T\hat{\mathcal{X}}$ and $X(x(t)) = X_H(x(t))$, $\forall t \in I$.

Take arbitrary $(\hat{Y}, \hat{\beta}) \in \hat{D}$. There exist $(Y, \beta) \in D$ such that $Y \sim_{\pi} \hat{Y}$ and $\beta = \pi^* \hat{\beta}$. Let $\hat{x}(t) = \pi(x(t))$, then

$$\begin{aligned} (\langle d\hat{H}, \hat{Y} \rangle + \langle \hat{\beta}, \hat{X} \rangle)(\hat{x}(t)) &= (\langle dH, Y \rangle + \langle \beta, X \rangle)(x(t)) \\ &= (\langle dH, Y \rangle + \langle \beta, X_H \rangle)(x(t)) = 0, \end{aligned}$$

where in the last step we used that $(X_H, dH)(x(t)) \in D(x(t)) = D^{\perp}(x(t)) = [D(x(t))]^{\perp}$, $\forall t \in I$. This shows that

$$(d\hat{H}, X_{\hat{H}})(\hat{x}(t)) \in [\hat{D}(\hat{x}(t))]^{\perp} = \hat{D}^{\perp}(\hat{x}(t)) = \hat{D}(\hat{x}(t)), \forall t \in I,$$

where we wrote $X_{\hat{H}}$ for \hat{X} . From the fact that $X \sim_{\pi} \hat{X}$ it follows that $\hat{x}(t) = X_{\hat{H}}(\hat{x}(t))$, $\forall t \in I$, so $\hat{x}(t)$ is a solution of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$. \square

REMARK 7. In general not every solution of (\mathcal{X}, D, H) projects to a solution of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$. Indeed, let $x(t)$ be a solution of (\mathcal{X}, D, H) , corresponding to the projectable vector field X_H , such that $x(t)$ projects to a solution $\hat{x}(t)$ of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$. Then the integral curve $y(t)$ corresponding to the vector field $X_H + Z$, where $Z \in \mathcal{G}_0 \cap T\mathcal{X}_c$, will also be a solution of (\mathcal{X}, D, H) . However, since Z is in general not projectable to a vector field on \mathcal{X}/G , $y(t)$ will not project to a solution of $\hat{x}(t)$ of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$.

When Assumption 4 is satisfied, it can be shown that the unique solution $x(t)$ will always project to a solution of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$, see Section 7.

Now, after these preliminaries, we are ready to investigate what is going to be the main result of our work.

5. The main result

In this section we will derive our main result on reduction of implicit generalized Hamiltonian systems. This result will generalize the “classical” reduction theorems of explicit Hamiltonian systems described in [1, 13, 15, 19].

Consider an implicit generalized Hamiltonian system (\mathcal{X}, D, H) on an n -dimensional manifold \mathcal{X} , with generalized Dirac structure D and Hamiltonian function $H \in C^\infty(\mathcal{X})$. Suppose the system has r independent first integrals $P_i \in C^\infty(\mathcal{X})$, $i = 1, \dots, r$, and suppose there exist corresponding independent vector fields $X_{P_i} \in T\mathcal{X}$, i.e. $(X_{P_i}, dP_i) \in D$, $i = 1, \dots, r$, such that each X_{P_i} is a strong symmetry of (\mathcal{X}, D, H) (in Section 8 these symmetries will be called *horizontal*). We assume that P_i and X_{P_i} satisfy the conditions

$$\{P_i, P_j\}_D = \sum_{k=1}^r c_{ij}^k P_k, \quad (40)$$

and

$$[X_{P_i}, X_{P_j}] = \sum_{k=1}^r c_{ij}^k X_{P_k}, \quad (41)$$

where $c_{ij}^k \in \mathbb{R}$ are constants, $i, j = 1, \dots, r$.

REMARK 8. Note that in the case of a Poisson structure on \mathcal{X} , which satisfies the Jacobi identity (i.e. which is closed), (40) implies (41) (in the case of a symplectic structure on \mathcal{X} , (40) and (41) are equivalent). However, in the case of a Dirac structure (i.e. which is closed) on \mathcal{X} , (40) implies only $[X_{P_i}, X_{P_j}] = \sum_{k=1}^r c_{ij}^k X_{P_k} + Z_{ij}$, where $Z_{ij} \in \mathcal{G}_0$.

Because of condition (41) there exists an r -dimensional Lie group G with the corresponding Lie algebra \mathcal{G} for which the infinitesimal generators $(\xi_i)_\mathcal{X} = X_{P_i}$, $i = 1, \dots, r$, where $\{\xi_1, \dots, \xi_r\}$ is a basis of \mathcal{G} [19]. It follows that G is a strong symmetry Lie group of (\mathcal{X}, D, H) . Let $\{\mu_1, \dots, \mu_r\}$ be a basis of \mathcal{G}^* . We define the following map from \mathcal{X} to \mathcal{G}^* , also called the *momentum map* [19, 1, 13],

$$P(x) = \sum_{i=1}^r P_i(x) \mu_i.$$

PROPOSITION 22. *The momentum map P is Ad^* -equivariant, that is,*

$$P(\phi_g(x)) = \text{Ad}_g^*(P(x)),$$

for all $x \in \mathcal{X}$, $g \in G$, where Ad^* is the coadjoint action corresponding to the Lie group G .

Proof: The proof equals the proof in [19], see also [1, 13], we only have to consider the bracket of admissible functions $\{\cdot, \cdot\}_D$ instead of the Poisson bracket $\{\cdot, \cdot\}$. \square

REMARK 9. The setup given above, originally used in [19] for explicit Hamiltonian systems defined on Poisson manifolds, is a little bit different than the setup used in [1, 13, 15]. Analogously to [1, 13, 15], it can be assumed that the implicit generalized Hamiltonian system (\mathcal{X}, D, H) admits a strong symmetry Lie group G , and a corresponding Ad^* -equivariant momentum map $P : \mathcal{X} \rightarrow \mathcal{G}^*$ such that

$$(\xi_{\mathcal{X}}, dP_{\xi}) \in D, \quad \forall \xi \in \mathcal{G}, \quad (42)$$

where $P_{\xi} \in C^{\infty}(\mathcal{X})$ is defined by $P_{\xi}(x) = P(x)(\xi)$, $\forall x \in \mathcal{X}$. Although the setup we choose above is slightly less general than in this remark, it does not make any difference for the results in the sequel.

Now we will describe the reduction possibilities of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) admitting the strong symmetry Lie group G corresponding to the first integrals P_1, \dots, P_r . There are two ways, which in a sense are dual, to reduce the Hamiltonian system. The first one is to begin by reducing the Hamiltonian system to a level set $P^{-1}(\mu)$ of the first integrals, using Proposition 19. At this point the resulting implicit generalized Hamiltonian system will have some symmetry remaining from the symmetry group G , however, in general it will not be the whole group G but only a subgroup G_{μ} of G . Then we can use Proposition 21 to further reduce the Hamiltonian system to an implicit generalized Hamiltonian system on the quotient manifold $P^{-1}(\mu)/G_{\mu}$. The second way to reduce the implicit generalized Hamiltonian system (\mathcal{X}, D, H) is by beginning to reduce the Hamiltonian system to an implicit generalized Hamiltonian system on the quotient manifold \mathcal{X}/G , as in Proposition 21. The resulting Hamiltonian system will have some first integrals (actually these will be Casimir functions) remaining from P_1, \dots, P_r which we can use to further reduce the Hamiltonian system to a level set of these first integrals, Proposition 19. The main result of our work will state that these two ways of reducing the implicit generalized Hamiltonian system (\mathcal{X}, D, H) will result in the same reduced implicit generalized Hamiltonian system (up to isomorphism). This is a generalization of the classical reduction theorems of [16, 1, 13, 15, 19].

Reduction first using the first integrals, then a remaining symmetry group.

Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) with the corresponding independent first integrals P_1, \dots, P_r and strong symmetry Lie group G as described previously. Because P_1, \dots, P_r are first integrals, the solutions of (\mathcal{X}, D, H) will live on some level set $\tilde{\mathcal{X}} = \{x \in \mathcal{X} \mid P_1(x) = a_1, \dots, P_r(x) = a_r, (a_1, \dots, a_r) \in \mathbb{R}^r\}$, $\tilde{\mathcal{X}} \cap \mathcal{X}_c$ nonempty. Note that by using the momentum map P we can denote this level set by $\tilde{\mathcal{X}} = P^{-1}(\mu)$ for some $\mu \in \mathcal{G}^*$. Using Proposition 19, assuming that $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in \tilde{\mathcal{X}}$, is constant dimensional on $\tilde{\mathcal{X}}$, we can reduce the Hamiltonian system to an implicit generalized Hamiltonian system $(P^{-1}(\mu), \tilde{D}, \tilde{H})$ on $P^{-1}(\mu)$, where \tilde{D} is the generalized Dirac structure induced by D , and $\tilde{H} = \iota_1^* H$ is the Hamiltonian function on $P^{-1}(\mu)$, $\iota_1 : P^{-1}(\mu) \rightarrow \mathcal{X}$ being

the inclusion map. Consider the subgroup

$$G_\mu = \{g \in G \mid \text{Ad}_g^*(\mu) = \mu\}, \quad (43)$$

or equivalently (by equivariance of P)

$$G_\mu = \{g \in G \mid \phi_g(P^{-1}(\mu)) \subset P^{-1}(\mu)\}.$$

G_μ is a subgroup of G and therefore a Lie group itself.

LEMMA 23. G_μ is a strong symmetry Lie group of $(P^{-1}(\mu), \bar{D}, \bar{H})$.

Proof: Consider $\bar{X} = (\xi_\mu)_{P^{-1}(\mu)}$ for some $\xi_\mu \in \mathcal{G}_\mu$. Then \bar{X} is ι_1 -related to $X = (\xi_\mu)_X$. Now, let $(\bar{Y}, \bar{\beta}) \in \bar{D}$, then $\bar{Y} \sim_{\iota_1} Y$ and $\bar{\beta} = \iota_1^* \beta$, $(Y, \beta) \in D$, see (17). Then $L_{\bar{X}} \bar{Y} = [\bar{X}, \bar{Y}] \sim_{\iota_1} [X, Y] = L_X Y$. Furthermore, $L_{\bar{X}} \bar{\beta} = L_{\bar{X}} \iota_1^* \beta = \iota_1^* L_X \beta$. Now, X is a symmetry of D which means that $(L_X Y, L_X \beta) \in D$, and it follows that also $(L_{\bar{X}} \bar{Y}, L_{\bar{X}} \bar{\beta}) \in \bar{D}$, so \bar{X} is a symmetry of \bar{D} . Because $L_{\bar{X}} \bar{H}_1 = L_{\bar{X}} \iota_1^* H = \iota_1^* L_X H = 0$, \bar{X} is a strong symmetry of $(P^{-1}(\mu), \bar{D}, \bar{H})$. \square

G_μ is called the residual symmetry group. Now we can use Proposition 21 (in Theorem 27 we will show that the assumptions of Proposition 21 are satisfied) to further reduce the Hamiltonian system $(P^{-1}(\mu), \bar{D}, \bar{H})$ to an implicit generalized Hamiltonian system $(P^{-1}(\mu)/G_\mu, \hat{\bar{D}}, \hat{\bar{H}})$ on the quotient manifold $P^{-1}(\mu)/G_\mu$, where $\hat{\bar{D}}$ is the generalized Dirac structure induced by \bar{D} , and $\hat{\bar{H}}$ is the Hamiltonian function on $P^{-1}(\mu)/G_\mu$, with $\bar{H} = \hat{\bar{H}} \circ \pi_\mu$, where $\pi_\mu : P^{-1}(\mu) \rightarrow P^{-1}(\mu)/G_\mu$ is the projection map.

Reduction first using the symmetry group, then the remaining first integrals. Again, consider the same implicit generalized Hamiltonian system (\mathcal{X}, D, H) with the corresponding independent first integrals P_1, \dots, P_r and strong symmetry Lie group G as we started with in the reduction process described above. Contrary to starting with reduction to a level set of the first integrals, as we did above, we will now reduce the Hamiltonian system (\mathcal{X}, D, H) by first reducing it to the quotient manifold \mathcal{X}/G . Assume that $V + G_0$ and $D \cap E_q$ are constant dimensional. Using Proposition 21 this gives us an implicit generalized Hamiltonian system $(\mathcal{X}/G, \hat{D}, \hat{H})$ on \mathcal{X}/G , where \hat{D} is the generalized Dirac structure induced by D , and \hat{H} is the Hamiltonian function on \mathcal{X}/G , with $H = \hat{H} \circ \pi$. Here $\pi : \mathcal{X} \rightarrow \mathcal{X}/G$ is the projection map. Consider the quotient space $\hat{\mathcal{G}}^* = \mathcal{G}^*/G$ of coadjoint orbits \mathcal{O}_μ in \mathcal{G}^* , along with the projection map $\varpi : \mathcal{G}^* \rightarrow \hat{\mathcal{G}}^*$. A coadjoint orbit is defined as

$$\mathcal{O}_\mu = \{\text{Ad}_g^*(\mu) \mid g \in G\}, \quad \mu \in \mathcal{G}^*. \quad (44)$$

Throughout we assume that $\hat{\mathcal{G}}^*$ is a smooth manifold. Define the map $\hat{P} : \mathcal{X}/G \rightarrow \hat{\mathcal{G}}^*$ by [13]

$$\hat{P} \circ \pi = \varpi \circ P. \quad (45)$$

Then \hat{P} is a conserved quantity along solutions of $(\mathcal{X}/G, \hat{D}, \hat{H})$. Indeed, let $\hat{x}(t)$ be a solution of $(\mathcal{X}/G, \hat{D}, \hat{H})$. Then there exists (locally) a solution $x(t)$ of (\mathcal{X}, D, H) such that $\pi(x(t)) = \hat{x}(t)$, see Proposition 21. The corresponding vector fields are related, i.e. $X_H \sim_\pi X_{\hat{H}}$. Then

$$\begin{aligned} \langle d\hat{P}, X_{\hat{H}} \rangle(\hat{x}(t)) &= \langle \pi^* d\hat{P}, X_H \rangle(x(t)) = \langle d(\hat{P} \circ \pi), X_H \rangle(x(t)) \\ &= \langle d(\varpi \circ P), X_H \rangle(x(t)) = d\varpi(\langle dP, X_H \rangle)(x(t)) \\ &= 0, \end{aligned} \quad (46)$$

where the last step follows from the fact that P is a first integral of (\mathcal{X}, D, H) . Actually, \hat{P} is a Casimir function, because take arbitrary $\hat{H} \in C^\infty(\mathcal{X}/G)$, then \hat{H} corresponds to a G -invariant function $H \in C^\infty(\mathcal{X})$, by $H = \hat{H} \circ \pi$, for which again P will be a first integral, and so by (46) \hat{P} will be conserved along solutions of $(\mathcal{X}/G, \hat{D}, \hat{H})$. In Section 6 we will elaborate a bit more on the map \hat{P} . In particular we will show that “locally $d\hat{P} \in \hat{P}_0$ ”. Using Proposition 20 (see also Section 6) we can restrict the Hamiltonian system $(\mathcal{X}/G, \hat{D}, \hat{H})$ to an implicit generalized Hamiltonian system $(\hat{P}^{-1}(\hat{\mu}), \hat{\tilde{D}}, \hat{\tilde{H}})$ on a level set $\hat{P}^{-1}(\hat{\mu})$ of \hat{P} , for some $\hat{\mu} \in \hat{\mathcal{G}}^*$ (to be consistent with the procedure above we should take $\hat{\mu} = \varpi(\mu)$). Here $\hat{\tilde{D}}$ is the generalized Dirac structure induced by \hat{D} , $\hat{\tilde{H}} = \iota_2^* \hat{H}$ is the Hamiltonian function on $\hat{P}^{-1}(\hat{\mu})$ and $\iota_2 : \hat{P}^{-1}(\hat{\mu}) \rightarrow \mathcal{X}/G$ is the inclusion map.

Consider the two reduction procedures described above.

LEMMA 24. *There exists a diffeomorphism ψ from $P^{-1}(\mu)/G_\mu$ to $\hat{P}^{-1}(\hat{\mu})$, with $\hat{\mu} = \varpi(\mu)$, such that the following diagram commutes:*

$$\begin{array}{ccc} & \mathcal{X} & \\ \iota_1 \nearrow & & \searrow \pi \\ P^{-1}(\mu) & & \mathcal{X}/G \\ \pi_\mu \downarrow & & \uparrow \iota_2 \\ P^{-1}(\mu)/G_\mu & \xrightarrow{\psi} & \hat{P}^{-1}(\hat{\mu}) \end{array} \quad (47)$$

Proof: The proof is based on [13]. First we prove that there exists a diffeomorphism $\psi : P^{-1}(\mu)/G_\mu \rightarrow \pi(P^{-1}(\mu))$. Note that $P^{-1}(\mu)$ is a submanifold of \mathcal{X} , so $\pi(P^{-1}(\mu))$ makes sense and is a subspace of \mathcal{X}/G .

Now, define $\psi : P^{-1}(\mu)/G_\mu \rightarrow \pi(P^{-1}(\mu))$ as follows: Let $\hat{x} \in P^{-1}(\mu)/G_\mu$. There exists an $\bar{x} \in P^{-1}(\mu)$ such that $\pi_\mu(\bar{x}) = \hat{x}$. Define $\psi(\hat{x}) = \pi(\bar{x})$. To see that ψ is well defined, let $\bar{x}' \in P^{-1}(\mu)$ be another element such that $\pi_\mu(\bar{x}') = \hat{x}$. Then there exists a $g \in G_\mu$ such that $\phi_g(\bar{x}) = \bar{x}'$ and it follows that $\pi(\bar{x}) = \pi(\bar{x}')$, so

ψ is well defined. We have to prove that ψ is a diffeomorphism. The fact that ψ is surjective is trivial. Now, let $\hat{x}_1, \hat{x}_2 \in P^{-1}(\mu)/G_\mu$ be such that $\psi(\hat{x}_1) = \psi(\hat{x}_2)$. Then $\psi(\hat{x}_1) = \pi(\bar{x}_1)$ and $\psi(\hat{x}_2) = \pi(\bar{x}_2)$ for $\bar{x}_1, \bar{x}_2 \in P^{-1}(\mu)$ with $\pi_\mu(\bar{x}_1) = \hat{x}_1$ and $\pi_\mu(\bar{x}_2) = \hat{x}_2$. So $\pi(\bar{x}_1) = \pi(\bar{x}_2)$ and therefore there exists a $g \in G$ such that $\phi_g(\bar{x}_1) = \bar{x}_2$. From Ad^* -equivariance of P , Proposition 22, it follows that $g \in G_\mu$. Indeed,

$$\text{Ad}_g^*(\mu) = \text{Ad}_g^*(P(\bar{x}_1)) = P(\phi_g(\bar{x}_1)) = P(\bar{x}_2) = \mu,$$

and comparing with (43) gives that $g \in G_\mu$. But $\phi_g(\bar{x}_1) = \bar{x}_2$ for some $g \in G_\mu$ implies that $\pi_\mu(\bar{x}_1) = \pi_\mu(\bar{x}_2)$ and so $\hat{x}_1 = \hat{x}_2$. That means that ψ is injective. So ψ is bijective and because we assume that all maps are smooth it follows that ψ is a diffeomorphism.

Secondly, we prove that $\pi(P^{-1}(\mu)) = \hat{P}^{-1}(\hat{\mu})$. $\pi(P^{-1}(\mu)) \subset \hat{P}^{-1}(\hat{\mu})$ is easy and follows directly from (45). We prove the converse inclusion. Take an arbitrary $\hat{x} \in \hat{P}^{-1}(\hat{\mu}) \subset \mathcal{X}/G$ and let $x \in \mathcal{X}$ be such that $\pi(x) = \hat{x}$. Then by (45), $\varpi(\mu) = \hat{\mu} = \hat{P}(\pi(x)) = \varpi(P(x))$, which implies that $\mu \in \mathcal{O}_{P(x)}$, so there exists a $g \in G$ such that $\text{Ad}_g^*(P(x)) = \mu$ by (44). However, by Ad^* -equivariance of P this means that $P(\phi_g(x)) = \mu$, so $\phi_g(x) \in P^{-1}(\mu)$. Furthermore, $\pi(\phi_g(x)) = \pi(x) = \hat{x}$. This proves the converse inclusion. \square

REMARK 10. A nice interpretation of $\hat{P}^{-1}(\hat{\mu})$ is given in the fact that it is equivalent to the quotient space $P^{-1}(\mathcal{O}_\mu)/G$, as can be easily seen. Lemma 24 then states that $P^{-1}(\mu)/G_\mu$ is diffeomorphic to $P^{-1}(\mathcal{O}_\mu)/G$, which is the well known orbit reduction theorem [14].

DEFINITION 10. Let M and N be two manifolds, and let $\tau : M \rightarrow N$ be a diffeomorphism. Let D_M be a (generalized) Dirac structure on M and let D_N be a (generalized) Dirac structure on N . Then τ is called a *Dirac isomorphism* if

$$(X, \alpha) \in D_M \iff (\tau_* X, (\tau^*)^{-1} \alpha) \in D_N. \quad (48)$$

In this case we call D_M and D_N isomorphic, denoted by $D_M \cong D_N$.

REMARK 11. Let D_M and D_N be isomorphic. It is very easy to prove that D_M is closed if and only if D_N is closed.

Note that by (10) every symmetry $\phi : \mathcal{X} \rightarrow \mathcal{X}$ of a generalized Dirac structure D is a Dirac isomorphism.

Recall the two possible reduction procedures described above. The first one starts with the reduction of (\mathcal{X}, D, H) to a level set of the first integrals, and after factoring out the residual symmetry group it results in the implicit generalized Hamiltonian system $(P^{-1}(\mu)/G_\mu, \hat{D}, \hat{H})$. The second one starts with the reduction of (\mathcal{X}, D, H) by factoring out the symmetry group, and after restriction to the level set of the remaining Casimirs it results in the implicit generalized Hamiltonian

system $(\hat{P}^{-1}(\hat{\mu}), \hat{\bar{D}}, \hat{\bar{H}})$. In Lemma 24 it is shown that there exists a diffeomorphism $\psi : P^{-1}(\mu)/G_\mu \rightarrow \hat{P}^{-1}(\hat{\mu})$.

THEOREM 25. *ψ is a Dirac isomorphism. That is, $\hat{\bar{D}}$ and $\hat{\bar{D}}$ are isomorphic, $\hat{\bar{D}} \cong \hat{\bar{D}}$.*

Proof: First, notice that it is sufficient to prove that

$$(\hat{X}, \hat{\alpha}) \in \hat{\bar{D}} \implies (\psi_* \hat{X}, (\psi^*)^{-1} \hat{\alpha}) \in \hat{\bar{D}}. \quad (49)$$

Assume that (49) holds. Being Dirac structures, $\hat{\bar{D}}$ and $\hat{\bar{D}}$ are (pointwise) linear spaces. Define

$$\psi(\hat{\bar{D}}) := \{(\psi_* \hat{X}, (\psi^*)^{-1} \hat{\alpha}) \mid (\hat{X}, \hat{\alpha}) \in \hat{\bar{D}}\}.$$

Since ψ_* and ψ^* are linear mappings, $\psi(\hat{\bar{D}})$ is a linear space. By (49), $\psi(\hat{\bar{D}}) \subset \hat{\bar{D}}$. However, because ψ is a diffeomorphism, the map $(\psi_*, (\psi^*)^{-1} \cdot)$ is a bijection. Therefore,

$$\dim \psi(\hat{\bar{D}})(\bar{x}) = \dim \hat{\bar{D}}(\hat{x}) = \dim P^{-1}(\mu)/G_\mu = \dim \hat{P}^{-1}(\hat{\mu}) = \dim \hat{\bar{D}}(\bar{x}),$$

$\forall \bar{x} \in \hat{P}^{-1}(\hat{\mu})$, $\hat{x} = \psi^{-1}(\bar{x})$, and it follows that actually $\psi(\hat{\bar{D}}) = \hat{\bar{D}}$.

We prove (49). Suppose $(\hat{X}, \hat{\alpha}) \in \hat{\bar{D}}$, we prove that $(\psi_* \hat{X}, (\psi^*)^{-1} \hat{\alpha}) \in \hat{\bar{D}}^\perp = \hat{\bar{D}}$.

The pair $(\hat{X}, \hat{\alpha}) \in \hat{\bar{D}}$ corresponds to the pairs

- $(\bar{X}, \bar{\alpha}) \in \bar{D}$ with $\bar{X} \sim_{\pi_\mu} \hat{X}$, $\bar{\alpha} = \pi_\mu^* \hat{\alpha}$,
- $(X, \alpha) \in D$ with $\bar{X} \sim_{\iota_1} X$, $\bar{\alpha} = \iota_1^* \alpha$.

Now, take an arbitrary pair $(\hat{Y}, \hat{\beta}) \in \hat{\bar{D}}$. This corresponds to the pairs

- $(\bar{Y}, \bar{\beta}) \in \bar{D}$ with $\bar{Y} \sim_{\iota_2} \hat{Y}$, $\bar{\beta} = \iota_2^* \hat{\beta}$,
- $(Y, \beta) \in D$ with $\bar{Y} \sim_\pi Y$, $\bar{\beta} = \pi^* \beta$.

For an arbitrary $\bar{x} \in \hat{J}^{-1}(\hat{\mu})$ we calculate

$$\langle (\psi^*)^{-1} \hat{\alpha}, \hat{Y} \rangle(\bar{x}) + \langle \hat{\beta}, \psi_* \hat{X} \rangle(\bar{x}). \quad (50)$$

First we work out the first term in the above equation. By definition

$$\langle (\psi^*)^{-1} \hat{\alpha}, \hat{Y} \rangle(\bar{x}) = \langle \hat{\alpha}(\hat{x}), T_{\bar{x}} \psi^{-1} \cdot \hat{Y}(\bar{x}) \rangle, \quad (51)$$

where $\hat{x} = \psi^{-1}(\bar{x})$. Now, $T_{\bar{x}} \psi^{-1} \cdot \hat{Y}(\bar{x})$ is a tangent vector to $J^{-1}(\mu)/G_\mu$ at the point \hat{x} . Because π_μ and therefore $T\pi_\mu$ is surjective, there exists a point $\bar{x} \in J^{-1}(\mu)$ such that $\pi_\mu(\bar{x}) = \hat{x}$, and a tangent vector $\bar{Z}(\bar{x}) \in T_{\bar{x}} J^{-1}(\mu)$ such that

$$T_{\bar{x}} \psi^{-1} \cdot \hat{Y}(\bar{x}) = T_{\bar{x}} \pi_\mu \cdot \bar{Z}(\bar{x}). \quad (52)$$

Then (51) becomes

$$\begin{aligned} \langle (\psi^*)^{-1} \hat{\alpha}, \hat{Y} \rangle(\tilde{x}) &= \langle \hat{\alpha}(\hat{x}), T_{\tilde{x}} \pi_{\mu} \cdot \bar{Z}(\tilde{x}) \rangle \\ &= \langle \bar{\alpha}(\tilde{x}), \bar{Z}(\tilde{x}) \rangle \\ &= \langle \alpha(x), T_{\tilde{x}} \iota_1 \cdot \bar{Z}(\tilde{x}) \rangle, \end{aligned} \quad (53)$$

where $x = \iota_1(\tilde{x})$.

Because ψ is a diffeomorphism, $T\psi^{-1} = (T\psi)^{-1}$ is invertible. Then (52) becomes $\tilde{Y}(\tilde{x}) = T_{\tilde{x}} \psi \cdot T_{\tilde{x}} \pi_{\mu} \cdot \bar{Z}(\tilde{x})$. This implies

$$\hat{Y}(\iota_2(\tilde{x})) = T_{\tilde{x}} \iota_2 \cdot \tilde{Y}(\tilde{x}) = T_{\tilde{x}} \iota_2 \cdot T_{\tilde{x}} \psi \cdot T_{\tilde{x}} \pi_{\mu} \cdot \bar{Z}(\tilde{x}) = T_x \pi \cdot T_{\tilde{x}} \iota_1 \cdot \bar{Z}(\tilde{x}),$$

where we used the commutativity of diagram (47), $\iota_2 \circ \psi \circ \pi_{\mu} = \pi \circ \iota_1$, which implies $T\iota_2 \circ T\psi \circ T\pi_{\mu} = T\pi \circ T\iota_1$. Since $Y \sim_{\pi} \hat{Y}$, it follows that $T_x \pi \cdot T_{\tilde{x}} \iota_1 \cdot \bar{Z}(\tilde{x}) = T_x \pi \cdot Y(x)$ (note that again by commutativity $\iota_2(\tilde{x}) = \pi(x)$) which implies that

$$T_{\tilde{x}} \iota_1 \cdot \bar{Z}(\tilde{x}) = Y(x) + Y_0(x), \quad (54)$$

where $Y_0(x) \in \ker T_x \pi$. Plugging (54) into (53) gives

$$\langle (\psi^*)^{-1} \hat{\alpha}, \hat{Y} \rangle(\tilde{x}) = \langle \alpha(x), Y(x) + Y_0(x) \rangle. \quad (55)$$

However, $\alpha(x)$ maps $\ker T_x \pi$ to zero. Indeed, $\ker T\pi = \text{span}_{C^{\infty}(X)} \{X_{P_j}\}$, i.e. the distribution spanned by the symmetry vector fields, and

$$\left\langle \alpha(x), \sum f_j(x) X_{P_j}(x) \right\rangle = - \left\langle \sum f_j(x) dP_j(x), X(x) \right\rangle = 0,$$

where we used that $(X, \alpha) \in D$ and $(X_{P_j}, dP_j) \in D$, and $\bar{X} \sim_{\iota_1} X$ which gives that $\langle dP_j(x), X(x) \rangle = 0$. Then (55) becomes

$$\langle (\psi^*)^{-1} \hat{\alpha}, \hat{Y} \rangle(\tilde{x}) = \langle \alpha(x), Y(x) \rangle. \quad (56)$$

Now we will work out the second term of (50), which is a bit easier.

$$\begin{aligned} \langle \hat{\beta}, \psi_* \hat{X} \rangle(\tilde{x}) &= \langle \hat{\beta}(\iota_2(\tilde{x})), T_{\tilde{x}} \iota_2 \cdot \psi_* \hat{X}(\tilde{x}) \rangle \\ &= \langle \hat{\beta}(\iota_2(\tilde{x})), T_{\tilde{x}} \iota_2 \cdot T_{\tilde{x}} \psi \cdot T_{\tilde{x}} \pi_{\mu} \cdot \bar{X}(\tilde{x}) \rangle \\ &\quad \text{and now using commutativity gives} \\ &= \langle \hat{\beta}(\iota_2(\tilde{x})), T_x \pi \cdot T_{\tilde{x}} \iota_1 \cdot \bar{X}(\tilde{x}) \rangle \\ &= \langle \pi^* \hat{\beta}(x), T_{\tilde{x}} \iota_1 \cdot \bar{X}(\tilde{x}) \rangle \\ &= \langle \beta(x), X(x) \rangle. \end{aligned} \quad (57)$$

Using Eqs. (56) and (57), Eq. (50) becomes

$$\langle (\psi^*)^{-1} \hat{\alpha}, \hat{Y} \rangle(\hat{x}) + \langle \hat{\beta}, \psi_* \hat{X} \rangle(\hat{x}) = \langle \alpha(x), Y(x) \rangle + \langle \beta(x), X(x) \rangle = 0, \quad (58)$$

because $(X, \alpha), (Y, \beta) \in D$. Note that $(\hat{Y}, \hat{\beta}) \in \hat{D}$ and $\hat{x} \in \hat{J}^{-1}(\hat{\mu})$ were arbitrarily chosen, so (58) proves that $(\psi_* \hat{X}, (\psi^*)^{-1} \hat{\alpha}) \in \hat{D}^\perp = \hat{D}$. This ends the proof. \square

Using Theorem 25 we can prove that the two reduced implicit generalized Hamiltonian systems $(P^{-1}(\mu)/G_\mu, \hat{D}, \hat{H})$ and $(\hat{P}^{-1}(\hat{\mu}), \hat{D}, \hat{H})$ are equivalent up to isomorphism. More precisely, we define two implicit generalized Hamiltonian systems to be isomorphic in the following sense.

DEFINITION 11. Consider two implicit generalized Hamiltonian systems (M, D_M, H_M) and (N, D_N, H_N) . We call the two systems *isomorphic* if there exists a diffeomorphism $\tau : M \rightarrow N$ such that τ is a Dirac isomorphism, i.e. D_M and D_N are isomorphic, and $H_M = H_N \circ \tau$.

The solutions of two isomorphic implicit generalized Hamiltonian systems are related by the diffeomorphism τ . This means that two isomorphic systems generate the same (up to a diffeomorphism) dynamic behaviour.

PROPOSITION 26. Consider two isomorphic (by some diffeomorphism $\tau : M \rightarrow N$) implicit generalized Hamiltonian systems (M, D_M, H_M) and (N, D_N, H_N) . Then, $x(t)$ is a solution of (M, D_M, H_M) if and only if $\tau(x(t))$ is a solution of (N, D_N, H_N) .

Proof: First notice that τ being a Dirac isomorphism implies that τ is pointwise an isomorphism between the two linear spaces $D_M(x)$ and $D_N(\tau(x))$. Let $x(t)$ be a solution of (M, D_M, H_M) , i.e. $(X_{H_M}, dH_M)(x(t)) \in D_M(x(t))$, $\forall t \in I$, where $X_{H_M}(x(t)) = \dot{x}(t)$, $\forall t \in I$. Because τ is pointwise an isomorphism it follows that, using $H_M = H_N \circ \tau$,

$$(X_{H_N}, dH_N)(\tau(x(t))) \in D_N(\tau(x(t))), \quad \forall t \in I,$$

where we defined

$$X_{H_N}(\tau(x(t))) = T_{x(t)}\tau \cdot X_{H_M}(x(t)), \quad \forall t \in I. \quad (59)$$

Because of (59) it follows that $\frac{d}{dt}\tau(x(t)) = X_{H_N}(\tau(x(t)))$, $\forall t \in I$, which implies that $\tau(x(t))$ is a solution of (N, D_N, H_N) . The converse statement is proven in the same way. \square

Finally, we come to the main result of this paper.

THEOREM 27. Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) . Suppose the system has r independent first integrals P_1, \dots, P_r , satisfying (40), and corresponding independent vector fields X_{P_1}, \dots, X_{P_r} , satisfying (41), which generate a strong symmetry Lie group G of (\mathcal{X}, D, H) . Assume that $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in$

$P^{-1}(\mu)$, is constant dimensional on $P^{-1}(\mu)$, and that $V + G_0$ and $D \cap E_q$ are constant dimensional on \mathcal{X} . Then, using the two reduction procedures described above, the implicit generalized Hamiltonian system (\mathcal{X}, D, H) reduces to implicit generalized Hamiltonian systems on the manifolds $P^{-1}(\mu)$, $P^{-1}(\mu)/G_\mu$, \mathcal{X}/G and $\hat{P}^{-1}(\hat{\mu})$ in diagram (47). The two implicit generalized Hamiltonian systems $(P^{-1}(\mu)/G_\mu, \bar{D}, \bar{H})$ and $(\hat{P}^{-1}(\hat{\mu}), \hat{\bar{D}}, \hat{\bar{H}})$ are isomorphic (by the diffeomorphism ψ given in diagram (47)).

Proof: Because $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in P^{-1}(\mu)$, is constant dimensional on $P^{-1}(\mu)$, the system (\mathcal{X}, D, H) can be reduced to the implicit generalized Hamiltonian system $(P^{-1}(\mu), \bar{D}, \bar{H})$, using Proposition 17. Let \bar{V}_μ denote the distribution on $P^{-1}(\mu)$ spanned by the infinitesimal generators of G_μ . Let \bar{G}_0 be the distribution as defined in Section 2 corresponding to the generalized Dirac structure \bar{D} . Finally, let \bar{E}_q be the bundle as defined in (27) corresponding to $P^{-1}(\mu)$. We show that constant dimensionality of $V + G_0$ and $D \cap E_q$ on \mathcal{X} implies constant dimensionality of $\bar{V}_\mu + \bar{G}_0$ and $\bar{D} \cap \bar{E}_q$ on $P^{-1}(\mu)$.

First note that $\bar{V}_\mu \subset \bar{G}_0$, because take arbitrary $\bar{X} \in \bar{V}_\mu$, then $\bar{X} \sim_{\iota_1} X = \sum_i h_i X_{P_i}$, $h_i \in C^\infty(\mathcal{X})$, for some $X \in V$ (because G_μ is the Lie subgroup of symmetries of G that leave the level set $P^{-1}(\mu)$ invariant, i.e. that are tangent to this level set). Because $(X, \sum_i h_i dP_i) \in D$, this implies that

$$\left(\bar{X}, \iota_1^* \sum_i h_i dP_i \right) = \left(\bar{X}, \sum_i (h_i \circ \iota_1) d(P_i \circ \iota_1) \right) = \left(\bar{X}, \sum_i (h_i \circ \iota_1) \cdot 0 \right) = (\bar{X}, 0) \in \bar{D},$$

and so $\bar{X} \in \bar{G}_0$. Furthermore, by definition of \bar{D} , \bar{G}_0 consists of all $\bar{X} \in TP^{-1}(\mu)$ such that $\bar{X} \sim_{\iota_1} X \in G_1$ with $(X, \alpha) \in D$ such that $\iota_1^* \alpha = 0$. This means $\bar{X} \sim_{\iota_1} X \in V + G_0$ which implies that $\bar{X} \in \bar{V}_\mu + G_0|_{P^{-1}(\mu)}$ (note that if $X \in G_0$ then $\langle dP_i, X \rangle = 0$, $i = 1, \dots, r$, so X is tangent to the level set $P^{-1}(\mu)$). Concluding we get that

$$\bar{V}_\mu + \bar{G}_0 = \bar{G}_0 = \bar{V}_\mu + G_0|_{P^{-1}(\mu)},$$

where $G_0|_{P^{-1}(\mu)}$ denotes the set of all vector fields in G_0 restricted to $P^{-1}(\mu)$. Now, since $V + G_0$ is constant dimensional on \mathcal{X} , it follows that $\bar{V}_\mu + G_0|_{P^{-1}(\mu)}$ is constant dimensional on $P^{-1}(\mu)$. Thus, $\bar{G}_0 = \bar{V}_\mu + \bar{G}_0$ is constant dimensional on $P^{-1}(\mu)$. Since \bar{G}_0 and $\bar{V}_\mu + \bar{G}_0$ are constant dimensional it follows that also $\text{ann}(\bar{V}_\mu) \cap \bar{P}_1$ is constant dimensional on $P^{-1}(\mu)$, where \bar{P}_1 is the co-distribution corresponding to \bar{D} as defined in Section 2. From \bar{G}_0 and $\text{ann}(\bar{V}_\mu) \cap \bar{P}_1$ constant dimensional it immediately follows that also $\bar{D} \cap \bar{E}_q$ is constant dimensional on $P^{-1}(\mu)$. So the assumptions of Proposition 21 are satisfied and we can reduce the system $(P^{-1}(\mu), \bar{D}, \bar{H})$ further to the implicit generalized Hamiltonian system $(P^{-1}(\mu)/G_\mu, \hat{\bar{D}}, \hat{\bar{H}})$. This proves the first part of the theorem.

For the second part, Lemma 24 states that there exists a diffeomorphism ψ which makes the diagram (47) commuting, that is $\pi \circ \iota_1 = \iota_2 \circ \psi \circ \pi_\mu$. Take arbitrary $\hat{x} \in P^{-1}(\mu)/G_\mu$ and let $\bar{x} \in P^{-1}(\mu)$ be such that $\pi_\mu(\bar{x}) = \hat{x}$, then

$$\hat{H}(\hat{x}) = H \circ \iota_1(\bar{x}) = \hat{H} \circ \pi \circ \iota_1(\bar{x}) = \hat{H} \circ \iota_2 \circ \psi \circ \pi_\mu(\bar{x}) = \hat{H} \circ \iota_2 \circ \psi(\hat{x}) = \tilde{\hat{H}} \circ \psi(\hat{x}),$$

proving that $\hat{H} = \tilde{\hat{H}} \circ \psi$. Since by Theorem 25 the two generalized Dirac structures \hat{D} and $\tilde{\hat{D}}$ are isomorphic, it follows that the two implicit generalized Hamiltonian systems $(P^{-1}(\mu)/G_\mu, \hat{D}, \hat{H})$ and $(\hat{P}^{-1}(\hat{\mu}), \tilde{\hat{D}}, \tilde{\hat{H}})$ are isomorphic. \square

EXAMPLE 10. Consider the Dirac structure given in Example 1 (with D closed), and the Hamiltonian system (\mathcal{X}, D, H) corresponding to a function $H \in C^\infty(\mathcal{X})$. Assuming the conditions in Theorem 27 are satisfied, the system reduces to Hamiltonian systems on $P^{-1}(\mu)/G_\mu$ and $\hat{P}^{-1}(\hat{\mu})$. The corresponding Dirac structure \hat{D} , respectively $\tilde{\hat{D}}$, is again a symplectic structure on $P^{-1}(\mu)/G_\mu$, respectively $\hat{P}^{-1}(\hat{\mu})$ (for a proof of this see [5]). This example shows that Theorem 27 is a generalization of the classical (symplectic) reduction theorems described in [16, 1, 13].

EXAMPLE 11. Consider the Dirac structure given in Example 2 (with D closed), and the Hamiltonian system (\mathcal{X}, D, H) corresponding to a function $H \in C^\infty(\mathcal{X})$. Assuming the conditions in Theorem 27 are satisfied, the system reduces to Hamiltonian systems on $P^{-1}(\mu)/G_\mu$ and $\hat{P}^{-1}(\hat{\mu})$. The corresponding Dirac structure \hat{D} , respectively $\tilde{\hat{D}}$, is again a Poisson structure on $P^{-1}(\mu)/G_\mu$, respectively $\hat{P}^{-1}(\hat{\mu})$ (for a proof of this see [5]). This example shows that Theorem 27 is a generalization of the classical (Poisson) reduction theorems described in [15, 19]. Note that the reduced system on $P^{-1}(\mu)$ does not represent a classical Poisson system, but it is described by an implicit generalized Hamiltonian system, with a Dirac structure as the underlying geometric structure. This was already noticed in [9].

6. The Casimir function \hat{P}

In this section we will take a closer look at the map \hat{P} introduced in the second reduction procedure in the previous section. In particular we will show that “locally $d\hat{P} \in \hat{P}_0$ ”, which allows us to moderate the proof of Proposition 20 a little bit such that the result still holds in case $\bar{\mathcal{X}} = \hat{P}^{-1}(\hat{\mu})$ (as is the case in the reduction procedure in Theorem 27).

Recall that the momentum map was defined as $P : \mathcal{X} \rightarrow \mathcal{G}^*$

$$P(x) = \sum_{i=1}^r P_i(x) \mu_i, \quad (60)$$

where $\{\mu_1, \dots, \mu_r\}$ is a basis of \mathcal{G}^* , and P_1, \dots, P_r are the first integrals of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) . Define the quotient manifold

$\hat{\mathcal{G}}^* = \mathcal{G}^*/G$ of coadjoint orbits \mathcal{O}_μ in \mathcal{G}^* , and the corresponding projection map $\varpi : \mathcal{G}^* \rightarrow \hat{\mathcal{G}}^*$. Define the map $\hat{P} : \mathcal{X}/G \rightarrow \hat{\mathcal{G}}^*$ by

$$\hat{P} \circ \pi = \varpi \circ P, \quad (61)$$

where $\pi : \mathcal{X} \rightarrow \mathcal{X}/G$ is the projection map. Because \mathcal{G}^* is the dual of the Lie algebra $\mathcal{G} = T_e G$, \mathcal{G}^* is globally isomorphic to \mathbb{R}^r via some isomorphism $\varphi : \mathcal{G}^* \rightarrow \mathbb{R}^r$. Since $\hat{\mathcal{G}}^* = \mathcal{G}^*/G$ is a manifold (under the appropriate assumptions on G) it is locally diffeomorphic to \mathbb{R}^m , where m is the dimension of $\hat{\mathcal{G}}^*$, via some diffeomorphism $\hat{\varphi}_U : U \subset \hat{\mathcal{G}}^* \rightarrow \mathbb{R}^m$. Consider a local chart $(U, \hat{\varphi}_U)$ of $\hat{\mathcal{G}}^*$, then (61) implies

$$\hat{\varphi}_U \circ \hat{P} \circ \pi(x) = \hat{\varphi}_U \circ \varpi \circ \varphi^{-1} \circ \varphi \circ P(x), \quad \forall x \in W \subset \mathcal{X}, \quad (62)$$

where W is such that $\hat{P} \circ \pi(W) \subset U$. Now, since $\hat{\varphi}_U \circ \varpi \circ \varphi^{-1} : \mathbb{R}^r \rightarrow \mathbb{R}^m$ is a projection, it is a linear map and therefore it can be described by a matrix $[\text{Proj}] \in \mathbb{R}^{m \times r}$. Note that $\varphi \circ P$ is exactly the r -vector of first integrals, i.e. $\varphi \circ P(x) = [P_1(x), \dots, P_r(x)]^T$. Then (62) becomes

$$\begin{aligned} \hat{\varphi}_U \circ \hat{P} \circ \pi(x) &= [\text{Proj}][P_1(x), \dots, P_r(x)]^T \\ &= \begin{bmatrix} c_{11}P_1(x) + \dots + c_{1r}P_r(x) \\ \vdots \\ c_{m1}P_1(x) + \dots + c_{mr}P_r(x) \end{bmatrix}, \end{aligned} \quad (63)$$

for some constants $c_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, r$. Now, $\hat{\varphi}_U \circ \hat{P}$ defines the m -vector $\hat{\varphi}_U \circ \hat{P}(\hat{x}) = [\hat{P}_1(\hat{x}), \dots, \hat{P}_m(\hat{x})]^T$, where $\hat{P}_i \in C^\infty(W/G)$, $i = 1, \dots, m$. By (63) it follows that

$$\pi^* d\hat{P}_i = c_{i1}dP_1 + \dots + c_{ir}dP_r, \quad i = 1, \dots, m.$$

Now, take an arbitrary pair $(\hat{Y}, \hat{\beta}) \in \hat{D}$. Then

$$\begin{aligned} \langle d\hat{P}_i, \hat{Y} \rangle(\hat{x}) &= \langle \pi^* d\hat{P}_i, Y \rangle(x) = \left\langle \sum_{j=1}^r c_{ij} dP_j, Y \right\rangle(x) \\ &= -\left\langle \beta, \sum_{j=1}^r c_{ij} X_{P_j} \right\rangle(x) = -\langle \hat{\beta}, 0 \rangle(\hat{x}) = 0, \end{aligned} \quad (64)$$

$\forall \hat{x} \in W/G$, where $x \in W$, $\pi(x) = \hat{x}$, $i = 1, \dots, m$, since $Y \sim_\pi \hat{Y}$, $\beta = \pi^* \hat{\beta}$, with $(Y, \beta) \in D$, and $\sum_j c_{ij} X_{P_j} \sim_\pi 0$. So locally $d\hat{P}_i \in \text{ann } \hat{G}_1 = \hat{P}_0$, $i = 1, \dots, m$. (64) is what we meant saying that “locally $d\hat{P} \in \hat{P}_0$ ”.

Now consider the implicit generalized Hamiltonian system $(\mathcal{X}/G, \hat{D}, \hat{H})$ in the reduction procedure of Theorem 27. The map \hat{P} is a Casimir function by (46) (or more correctly, by (64)). As in Proposition 20 we want to conclude that the solutions of $(\mathcal{X}/G, \hat{D}, \hat{H})$ lying in $\hat{P}^{-1}(\hat{\mu})$ are exactly the solutions of the reduced system $(\hat{P}^{-1}(\hat{\mu}), \hat{\bar{D}}, \hat{\bar{H}})$. Since it is in general not true that $d\hat{P} \in \hat{P}_0$ we cannot use Proposition 20 directly. However, since (64) holds, and since the level set $\hat{P}^{-1}(\hat{\mu})$ is locally given by the level set of $\hat{P}_1, \dots, \hat{P}_m$, we can conclude that for every pair $(\hat{Y}, \hat{\beta}) \in \hat{D}$ it holds that \hat{Y} is tangent to $\hat{P}^{-1}(\hat{\mu})$. Then we can copy the rest of the proof of Proposition 20 to conclude that the solutions of $(\mathcal{X}/G, \hat{D}, \hat{H})$ lying in $\hat{P}^{-1}(\hat{\mu})$ are exactly the solutions of the reduced system $(\hat{P}^{-1}(\hat{\mu}), \hat{\bar{D}}, \hat{\bar{H}})$.

7. Implicit generalized Hamiltonian systems with index 1

In this section we take a closer look at the reduction procedure in Theorem 27 in case the implicit generalized Hamiltonian system (\mathcal{X}, D, H) satisfies Assumption 4. The motivation for this is as follows. Considering the reduction procedure in Theorem 27, notice that we have made some assumptions.

- (i) To define the generalized Dirac structure \bar{D} on the submanifold $P^{-1}(\mu)$, we needed the assumption that $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in P^{-1}(\mu)$, is constant dimensional on $P^{-1}(\mu)$.
- (ii) To define the generalized Dirac structure \hat{D} on the quotient manifold \mathcal{X}/G , we needed the assumption that $V + G_0$ and $D \cap E_q$ are constant dimensional on \mathcal{X} .
- (iii) Finally, concerning Proposition 21 about reduction of an implicit generalized Hamiltonian system (\mathcal{X}, D, H) to an implicit generalized Hamiltonian system $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$ on a quotient manifold $\hat{\mathcal{X}}$, we needed the assumption of projectability of a solution $x(t)$ to show that it reduces to a solution $\hat{x}(t)$ of $(\hat{\mathcal{X}}, \hat{D}, \hat{H})$.

These three assumptions are new with respect to the assumptions made in the classical reduction theorems of [16, 1, 13, 15, 19]. Indeed, considering the reduction of classical explicit Hamiltonian systems like in Examples 10 and 11, these three assumptions are void. For take an explicit Hamiltonian system defined with respect to a symplectic structure as in Example 10. Because $G_1 = T\mathcal{X}$ is constant dimensional, $G_1(\bar{x}) \cap T_{\bar{x}}\bar{\mathcal{X}} = T_{\bar{x}}\bar{\mathcal{X}}$, $\bar{x} \in \bar{\mathcal{X}}$, is constant dimensional on $\bar{\mathcal{X}}$, which implies that $D(\bar{x}) \cap E_s(\bar{x})$, $\bar{x} \in \bar{\mathcal{X}}$, is constant dimensional on $\bar{\mathcal{X}}$, see Remark 3. Also, since $G_0 = 0$, $V + G_0 = V$ is constant dimensional (with $\dim V = r = \dim G$). Furthermore, since $P_1 = T^*\mathcal{X}$, $\text{ann}(V) \cap P_1 = \text{ann}(V)$ is constant dimensional, and together with G_0 constant dimensional this implies that $D \cap E_q$ is constant dimensional on \mathcal{X} . Finally, the vector field $X_{\bar{H}} \in TP^{-1}(\mu)$, corresponding to a solution $\bar{x}(t)$ of $(P^{-1}(\mu), \bar{D}, \bar{H})$ coming from a solution $x(t)$ of (\mathcal{X}, D, H) , is projectable to a vector field on $P^{-1}(\mu)/G_\mu$ [16, 1, 13]. Note that the reduced Hamiltonian system $(\mathcal{X}/G, \hat{D}, \hat{H})$ on \mathcal{X}/G is not a symplectic system anymore, so the reduction procedures in [16, 1, 13] do not include the system $(\mathcal{X}/G, \hat{D}, \hat{H})$. However,

$(\mathcal{X}/G, \hat{D}, \hat{H})$ is a Poisson system, and in [15, 19] it is proved that every solution of (\mathcal{X}, D, H) projects to a solution of $(\mathcal{X}/G, \hat{D}, \hat{H})$.

With respect to the second classical example, consider an explicit Hamiltonian system defined on a Poisson structure as in Example 11. Just as in the symplectic case $G_0 = 0$ and $P_1 = T^*\mathcal{X}$ imply that $V + G_0$ and $D \cap E_q$ are constant dimensional on \mathcal{X} . Furthermore, in [15, 19] it is shown that every solution $x(t)$ of (\mathcal{X}, D, H) projects to a solution $\hat{x}(t)$ of $(\mathcal{X}/G, \hat{D}, \hat{H})$. Again note that the reduced Hamiltonian system $(P^{-1}(\mu), \bar{D}, \bar{H})$ on $P^{-1}(\mu)$ is not a Poisson system anymore, and therefore is not included in the reduction procedures in [15, 19]. Under assumption (i), the reduced system on $P^{-1}(\mu)$ can be described as an implicit generalized Hamiltonian system on $P^{-1}(\mu)$. In [9] it is shown that assumption (i) is equivalent to the condition that every point $\bar{x} \in P^{-1}(\mu)$ lies on a *principal orbit* (of the group action of G on \mathcal{X}).

We saw in Proposition 5 that, assuming the implicit generalized Hamiltonian system (\mathcal{X}, D, H) satisfies Assumption 4, the system can be reduced to an explicit generalized Hamiltonian system $(\mathcal{X}_c, D_c, H_c)$ given by (9) (where the generalized Dirac structure D_c is defined by the structure matrix J_c). Then considering the examples above we would expect that the assumptions (ii) and (iii) are again automatically satisfied (because (\mathcal{X}, D, H) is in essence the explicit system $(\mathcal{X}_c, D_c, H_c)$). Note that we already saw in the Poisson case that we cannot expect assumption (i) to be satisfied in general. Here we will investigate the contents of assumptions (ii) and (iii) if the system (\mathcal{X}, D, H) satisfies Assumption 4.

Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) and assume that Assumption 4 is satisfied. Assumption (ii) says that $V + G_0$ and $D \cap E_q$ should be constant dimensional. Since V is constant dimensional and by Assumption 4 also $G_0 = \ker P_1$ is constant dimensional $V + G_0$ will be constant dimensional as well if and only if $V \cap G_0$ is constant dimensional. Consider a strong symmetry X_{P_i} of (\mathcal{X}, D, H) , then by ([23], Proposition 17) X_{P_i} will be tangent to \mathcal{X}_c , so $X_{P_i}(x_c) \in T_{x_c}\mathcal{X}_c$, $\forall x_c \in \mathcal{X}_c$. Furthermore, by Assumption 4 it follows that $G_0(x_c) \cap T_{x_c}\mathcal{X}_c = 0$, $\forall x_c \in \mathcal{X}_c$, see also ([23], Proposition 17). Because V is the distribution spanned by the symmetries X_{P_i} , $i = 1, \dots, r$, which generate the Lie group G , we have that $V(x_c) \cap G_0(x_c) = 0$, $\forall x_c \in \mathcal{X}_c$, which implies that $V + G_0$ is constant dimensional on \mathcal{X}_c . Secondly, since P_1 is constant dimensional by Assumption 4, $\text{ann}(V + G_0) = \text{ann}(V) \cap P_1$. Now, $V + G_0$ constant dimensional on \mathcal{X}_c implies $\text{ann}(V) \cap P_1$ constant dimensional on \mathcal{X}_c and it follows that also $D \cap E_q$ is constant dimensional on \mathcal{X}_c .

Assumption (iii) says that a solution $x(t)$ of (\mathcal{X}, D, H) should be projectable in order to reduce to a solution $\hat{x}(t)$ of $(\mathcal{X}/G, \hat{D}, \hat{H})$. Take an arbitrary solution $x(t)$ of (\mathcal{X}, D, H) , i.e. $\dot{x}(t) = X_H(x(t))$ where $X_H(x_c) \in T_{x_c}\mathcal{X}_c$, $\forall x_c \in \mathcal{X}_c$, is the *unique* vector field on \mathcal{X}_c (by Assumption 4, i.e. the vector field corresponding to the explicit system (9)) corresponding to H . By ([23], Proposition 17)

$$[(\xi)_{\mathcal{X}}, X_H](x_c) = 0, \quad \forall x_c \in \mathcal{X}_c, \quad (65)$$

for all symmetries $(\xi)_{\mathcal{X}}$, where $\xi \in \mathcal{G}$. This implies that $[V, X_H](x_c) \in V(x_c)$, $\forall x_c \in \mathcal{X}_c$, which implies that X_H is projectable on \mathcal{X}_c to a vector field \hat{X} on \mathcal{X}_c/G . Using the smooth Tietze extension theorem we can extend X_H to a vector field $X \in T\mathcal{X}$ which is projectable to a vector field on \mathcal{X}/G .

Furthermore, a solution $\bar{x}(t)$ of $(P^{-1}(\mu), \bar{D}, \bar{H})$, coming from a solution $x(t)$ of (\mathcal{X}, D, H) , should be projectable in order to reduce to a solution $\hat{x}(t)$ of $(P^{-1}(\mu)/G_\mu, \hat{D}, \hat{H})$. Consider an arbitrary solution $x(t)$ of (\mathcal{X}, D, H) in $P^{-1}(\mu)$, i.e. $\dot{x}(t) = X_H(x(t))$. By Proposition 19, $X_{\bar{H}} \sim_{\iota_1} X_H$. Consider an arbitrary symmetry $(\xi)_{\bar{\mathcal{X}}} \in \bar{V}_\mu$, where $\xi \in \mathcal{G}_\mu$ (note that \mathcal{G}_μ is a Lie subalgebra of \mathcal{G}), then $(\xi)_{\bar{\mathcal{X}}} \sim_{\iota_1} (\xi)_{\mathcal{X}}$. Then by (65) it follows that

$$[(\xi)_{\bar{\mathcal{X}}}, X_{\bar{H}}](\bar{x}_c) = 0, \quad \forall \bar{x}_c \in \mathcal{X}_c \cap P^{-1}(\mu).$$

This implies that

$$[\bar{V}_\mu, X_{\bar{H}}](\bar{x}_c) \in \bar{V}_\mu(\bar{x}_c), \quad \forall \bar{x}_c \in \mathcal{X}_c \cap P^{-1}(\mu),$$

which implies that $X_{\bar{H}}$ is projectable on $\mathcal{X}_c \cap P^{-1}(\mu)$ to a vector field \hat{X} on $(\mathcal{X}_c \cap P^{-1}(\mu))/G_\mu$. Using the smooth Tietze extension theorem we can extend $X_{\bar{H}}$ to a vector field on $P^{-1}(\mu)$ which is projectable to a vector field on $P^{-1}(\mu)/G_\mu$. We conclude that the solutions of (\mathcal{X}, D, H) and $(P^{-1}(\mu), \bar{D}, \bar{H})$ all satisfy the projectability assumption.

The above results have the following interpretation. Consider the implicit generalized Hamiltonian system (\mathcal{X}, D, H) and assume that Assumption 4 is satisfied. Then the system reduces to the *explicit* generalized Hamiltonian system $(\mathcal{X}_c, D_c, H_c)$ given by (9). The solutions of the implicit system (\mathcal{X}, D, H) are *exactly* the solutions of the explicit system $(\mathcal{X}_c, D_c, H_c)$, so, like in the classical cases in Examples 10 and 11, they should *always* be projectable to solutions on the reduced systems. As we have shown above, this is indeed the case (assumption (iii) is always satisfied). On the other hand, however, we could *not* show that assumption (i) and (ii) are always satisfied. Indeed, even in the classical case of a Poisson structure on \mathcal{X} , we need assumption (i) to describe the reduced system on $P^{-1}(\mu)$ as an implicit generalized Hamiltonian system. Although for the explicit Hamiltonian system $(\mathcal{X}_c, D_c, H_c)$, so for the *reduced* generalized Dirac structure D_c , assumption (ii) is always satisfied, like in Examples 10 and 11, this is in general *not* the case for the *original* generalized Dirac structure D . We could only show that $V + G_0$ and $D \cap E_q$ are constant dimensional on \mathcal{X}_c .

8. Constrained mechanical systems

In this section we connect the theory described above to the theory of symmetries and reduction in nonholonomically constrained mechanical systems, as described for instance in [3, 7, 22, 8]. In particular we will define horizontal symmetries, which will give rise to conserved quantities, and give an analogue of the reduction procedure described in [22].

In Definition 5 we defined a symmetry of a generalized Dirac structure D on \mathcal{X} as a vector field $f \in T\mathcal{X}$ such that $(L_f X, L_f \alpha) \in D$ for all $(X, \alpha) \in D$. We define a *horizontal symmetry* as follows.

DEFINITION 12. Let D be a generalized Dirac structure. A *horizontal symmetry* of D is a symmetry f of D (as in Definition 5) such that $f \in G_1$.

Note that G_1 describes the set of admissible flows, denoted by the constrained distribution F in ([3, 22, 8]). Very important for the applicability of our theory is the following proposition, given in the context of constrained mechanical systems in [22].

PROPOSITION 28. Let D be a generalized Dirac structure and denote by G a symmetry Lie group of D . Consider the infinitesimal symmetries generated by G . The set of horizontal symmetries is generated by a normal Lie subgroup G_n of G .

Proof: Let $\xi_n \in \mathcal{G}$ generate a horizontal symmetry of D . Take an arbitrary element $\xi \in \mathcal{G}$, then $\xi_{\mathcal{X}}$ is a symmetry of D . Because $L_f G_1 \subset G_1$ for every symmetry f of D it follows that $[\xi_{\mathcal{X}}, (\xi_n)_{\mathcal{X}}] \in G_1$, i.e. is again a horizontal symmetry. Thus the elements in \mathcal{G} generating horizontal symmetries form an ideal \mathcal{G}_n in \mathcal{G} . This ideal \mathcal{G}_n defines a normal Lie subgroup G_n of G with Lie algebra \mathcal{G}_n . \square

Horizontal symmetries are very important because they give rise to first integrals. Assume that G is a strong symmetry Lie group of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) , Definition 9. In the sequel we will assume that G_1 is constant dimensional. Then, by Theorem 3, the generalized Dirac structure D can be written as

$$D = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \alpha(x) - \omega(x)X(x) \in \text{ann } G_1(x), \forall x \in \mathcal{X}, X \in G_1\},$$

where $\omega : G_1 \rightarrow (G_1)^*$ is a skew-symmetric linear map. We will now define the (horizontal) momentum map corresponding to a symmetry Lie group G of D . The definition will be consistent with the one given in Section 5. Let the 2-form $\omega_1 : T\mathcal{X} \rightarrow T^*\mathcal{X}$ be an extension of ω , i.e. $\omega_1|_{G_1} = \omega$, and assume that there exists an Ad^* -equivariant momentum map $P : \mathcal{X} \rightarrow \mathcal{G}^*$ for the action of G , i.e.

$$d(P, \xi) = \omega_1(\xi_{\mathcal{X}}, \cdot), \quad \forall \xi \in \mathcal{G}. \quad (66)$$

Notice that P is a conserved quantity for the unconstrained ($G_1 = T\mathcal{X}$) system (which is the classical Noether theorem), but in general this is not the case for the constrained system. However, the part of P corresponding to the horizontal symmetries will be conserved. Therefore, define the *horizontal momentum map* $P_{\text{hor}} : \mathcal{X} \rightarrow \mathcal{G}_n^*$ to be the restriction of P to \mathcal{G}_n . Let $\{\xi_n^1, \dots, \xi_n^s\}$ be a basis of \mathcal{G}_n , with dual basis $\{\mu_n^1, \dots, \mu_n^s\} \in \mathcal{G}_n^*$, then P_{hor} can be written as

$$P_{\text{hor}}(x) = \sum_{i=1}^s P_{\xi_n^i}(x) \mu_n^i,$$

where $P_{\xi_n^i}(x) = \langle P(x), \xi_n^i \rangle$, $i = 1, \dots, s$. Indeed, $((\xi_n^i)_\mathcal{X}, dP_{\xi_n^i}) \in D$, $i = 1, \dots, s$, so by Proposition 11 every $P_{\xi_n^i}$ is a first integral of the system. Furthermore, the horizontal momentum map inherits the Ad^* -equivariance from the Ad^* -equivariance of P . So we are in the situation described in Section 5 and we can perform the reduction described in Theorem 27. Notice that we do not require (40) to hold since all we actually need in Section 5 is the Ad^* -equivariance of the corresponding momentum map P_{hor} . For clarity we state the previous results in a proposition.

PROPOSITION 29. *Consider a strong symmetry Lie group G of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) (with G_1 constant dimensional). Assume that ω can be extended to a 2-form ω_1 such that there exists an Ad^* -equivariant momentum map $P : \mathcal{X} \rightarrow \mathcal{G}^*$ for the action of G , given by (66). Let G_n be the normal Lie subgroup of G corresponding to the horizontal symmetries. Define the horizontal momentum map $P_{\text{hor}} : \mathcal{X} \rightarrow \mathcal{G}_n^*$ to be the restriction of P to G_n . Then P_{hor} is a first integral for (\mathcal{X}, D, H) . Furthermore, the conditions in Section 5 are satisfied and we can perform the reduction procedure described in Theorem 27 (assuming the conditions on constant dimensionality are satisfied).*

If we apply the above result to constrained mechanical systems (where $\omega : T\mathcal{X} \rightarrow T^*\mathcal{X}$ is the canonical 2-form on $\mathcal{X} = T^*Q$, see Example 5), we obtain the reduction with conserved momenta [22], see also [8, 3].

Notice that the definition of the horizontal momentum map is not unique because it depends on the extension of ω . We will show however that the reduction described in Proposition 29, possibly after reduction to the level set of a Casimir function, will result in a uniquely defined implicit generalized Hamiltonian system. So assume that ω_1 is an extension of ω with corresponding horizontal momentum map denoted by P_{hor}^1 , as described above. Let the 2-form $\omega_2 : T\mathcal{X} \rightarrow T^*\mathcal{X}$ be a second extension of ω and assume that there exists an Ad^* -equivariant momentum map $P^2 : \mathcal{X} \rightarrow \mathcal{G}^*$ for the action of G , defined with respect to ω_2 as in (66). Denote the corresponding horizontal momentum map with $P_{\text{hor}}^2 : \mathcal{X} \rightarrow \mathcal{G}_n^*$. Define the function $C : \mathcal{X} \rightarrow \mathcal{G}_n^*$ by $C(x) = P_{\text{hor}}^1(x) - P_{\text{hor}}^2(x)$, $x \in \mathcal{X}$, and let $C_{\xi_n^i} := \langle C, (\xi_n^i)_\mathcal{X} \rangle = P_{\xi_n^i}^1 - P_{\xi_n^i}^2$, $i = 1, \dots, s$. Then

$$dC_{\xi_n^i}|_{G_1} = (dP_{\xi_n^i}^1 - dP_{\xi_n^i}^2)|_{G_1} = (\omega_1((\xi_n^i)_\mathcal{X}, \cdot) - \omega_2((\xi_n^i)_\mathcal{X}, \cdot))|_{G_1} = 0,$$

since $\omega_1|_{G_1} = \omega_2|_{G_1} = \omega$, $i = 1, \dots, s$. This implies that C (or rather $C_{\xi_n^i}$, $i = 1, \dots, s$) is a Casimir function of (\mathcal{X}, D, H) . Restrict the system to a level set of C , i.e. $c_0 = C(x) = P_{\text{hor}}^1(x_0) - P_{\text{hor}}^2(x_0) = \mu_0^1 - \mu_0^2$, determined by the initial condition x_0 , and denote the resulting implicit generalized Hamiltonian system by $(\mathcal{X}_C, D_C, H_C)$, where $\mathcal{X}_C = C^{-1}(c_0)$. Notice that G_n leaves \mathcal{X}_C invariant because C is a Casimir function, and will be a strong symmetry Lie group of $(\mathcal{X}_C, D_C, H_C)$. Restrict the functions P_{hor}^1 and P_{hor}^2 to \mathcal{X}_C . Then, since $P_{\text{hor}}^1(x) - P_{\text{hor}}^2(x) = c_0$ is a constant,

$$P_{\mu_0^1} := (P_{\text{hor}}^1)^{-1}(\mu_0^1) = (P_{\text{hor}}^2)^{-1}(\mu_0^2) =: P_{\mu_0^2},$$

and by equivariance of both maps

$$G_n^{\mu_0^1} = \{g \in G_n \mid \phi_g(P_{\mu_0^1}) \subset P_{\mu_0^1}\} = \{g \in G_n \mid \phi_g(P_{\mu_0^2}) \subset P_{\mu_0^2}\} = G_n^{\mu_0^2},$$

i.e. both residual symmetry groups are equal. Therefore reduction of the implicit generalized Hamiltonian system $(\mathcal{X}_C, D_C, H_C)$ to $P_{\mu_0^1}/G_n^{\mu_0^1}$ will equal the reduction to $P_{\mu_0^2}/G_n^{\mu_0^2}$. Concluding we have the following proposition.

PROPOSITION 30. *The reduction described in Proposition 29, possibly after reduction to the level set of a Casimir function, does not depend on the extension of ω . That is, the resulting implicit generalized Hamiltonian system will be the same for every extension of ω chosen.*

Propositions 29 and 30 describe the reduction of the implicit generalized Hamiltonian system (\mathcal{X}, D, H) using the horizontal symmetries generated by G_n , and using the fact that these symmetries give rise to first integrals. This reduces the dimension of the dynamics by $\dim G_n + \dim G_n^{\mu_0}$. It can be proved that the resulting implicit generalized Hamiltonian system will still have the strong symmetry Lie group G/G_n left (see [22] for the case of constrained mechanical systems). Notice that the symmetries generated by G/G_n in general will not give rise to first integrals. Then we can use Proposition 21, assuming the conditions are satisfied, to further reduce the system, which will give another reduction of the dimension of the dynamics by $\dim G/G_n$. So in the end we have reduced the dimension of the dynamics by $\dim G + \dim G_n^{\mu_0}$. See [22] for the introduction of this idea in constrained mechanical systems.

9. Conclusions

In this paper we have extended the reduction theory for explicit Hamiltonian systems and kinematically constrained mechanical systems to a general reduction theory for implicit generalized Hamiltonian systems. We started with studying the notion of symmetry for implicit generalized Hamiltonian systems, as defined in [11, 23]. We derived some basic results on symmetries and introduced the notions of first integral (or conserved quantity) and Casimir function in this setting. The main part of the paper involves the study of the reduction of implicit generalized Hamiltonian systems. We showed that implicit generalized Hamiltonian systems can be reduced to systems on submanifolds, e.g. in the case of a level set of a first integral, or quotient manifolds, e.g. in the case of factoring out a strong symmetry Lie group of the system, in both cases giving rise to a reduced system which is again an implicit generalized Hamiltonian system. We combined these results to describe the reduction process in case the implicit generalized Hamiltonian system admits a strong symmetry Lie group with corresponding first integrals. We showed that reducing the system by starting with restriction to the level set of the conserved quantities and then factoring out the (residual) symmetry group, or first factoring out the symmetry

group and then restricting to the level set of the remaining conserved quantities results in the same (up to isomorphism) implicit generalized Hamiltonian system. This result generalizes the classical reduction theorems of explicit Hamiltonian systems as described in [16, 1, 15, 13, 19]. Furthermore, we related our results to the theory of symmetries and reduction in constrained mechanical systems [3, 7, 8, 22, 23] (which can also be described as implicit generalized Hamiltonian systems), giving the analogue of the reduction process described in [22].

The general setting, using the geometric notion of a Dirac structure and corresponding implicit generalized Hamiltonian systems, makes the theory applicable not only to mechanical systems with nonholonomic constraints, but to any multibody system, as well as to electromechanical systems (see e.g. [24]).

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